SPARE PARTS PROVISIONING FOR

ROTATABLE, FLEET-OPERATED COMPONENTS

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SUMMARY

This investigation was concerned with the extension of the bounds of knowledge relating to the design of maintenance facilities and spares-stocking policies for systems which are essentially characterized by the use of rotatable, fleet-operated components. The general research objective was the development of a theoretical basis for the representation and study of congestion phenomena associated with the flow of nonoperative components through a limited-capacity service facility and their subsequent storage before being returned to use.

The models of interest were termed "repairman models with spares" because of their structural similarity to the classical "repairman models" or, equivalently, "machine interference models" of queueing theory. Both regular repairman systems and those with spares have fixed numbers of service channels and working positions for units. However, regular repairman systems have a number of circulating units precisely equal to the number of working positions, while repairman systems with spares have a number of units which exceeds the number of working positions.

An initial concern was the finding of a methodological background to support the sparse offerings of the few published papers identifiable as treating repairman problems with spares. It was found that the literature on repairman models--the direct historical and natural methodological precursors to repairman models with spares--provides a rich source of theory, analytic techniques, and procedural philosophy

of potential usefulness to any study in the realm of repairman models with spares.

By contrasting the existing bounds of knowledge on the two types of models, it was possible to obtain a measure of the relative level of development of theory on repairman models with spares. The existing, significant repairman models with spares, namely the models of Taylor and Jackson (1954) and Toft and Boothroyd (1959), were seen to be analogous to the fundamental repairman model devised by Palm (1947). Even the limited level of theoretical sophistication suggested by these simple models was diminished when it was shown in the present investigation that both models (and Palm's as well) follow readily from an appropriate specification of parameters in a well-known, general queueing model.

The theoretical background supporting the study of repairman models with spares was further augmented with the introduction of topics from queueing network theory. In ascending order of complexity and generality, the hierarchy of models discussed was: (1) repairman models, (2) repairman models with spares, (3) cyclic queues, (4) finite queues, and (5) networks of queues. Conceptually, each group of models may be viewed as being generalizations of models lower in the hierarchy and as being specializations of models higher in the hierarchy.

Two sections of the thesis were devoted to the analytical development of new repairman models with spares. The first section treated Poisson repairman models with spares, that is, models in which the component failure times and service times follow independent, one-parameter, negative-exponential probability distributions. Topics treated were: spares which fail in storage, extended service intervals, time-dependent conditions, servicing with ancillary functions, faulty repairs, transit delays, multiple failure modes, and multiple types of service. One or more repairman models with spares were derived in association with each topic. The principal methods of attack were those of basic Poisson queueing theory and those of cyclic and finite queueing network theory.

The second section treated repairman models with spares in which the component failure times were negative-exponentially distributed and the component service times followed a two-parameter, scale-modified chi-square distribution (i.e., an "Erlang distribution"). The key to formulating models in this section was the decomposition of chi-square distributed phenomena through a "method of successive stages" into several negative-exponentially distributed phases. The steady-state results obtained were of a surprisingly simple form in view of the complex manipulations required to obtain them. It was observed that the schemes used in formulation of these models had general applicability and their extension to the formulation of more complex repairman models with spares was discussed.

Generally, the models obtained were a threefold representation of a system of interest: (1) a set of differential-difference equations relating the probabilities $P_n(t)$ of being in various states of the system at time t (the "time-variant" or "dynamic" model); a set of difference equations relating the proportions of time p_n spent in each state after the system has been operating for a long time (the "steady-state" or "stationary" model); and (3) explicit formulae for the p_n

(the stationary solution). However, in one instance explicit formulae were developed for the $P_n(t)$ and the prospects for obtaining time-dependent solutions to other dynamic models were discussed.

In most cases, model results were given in a form quite tractable to hand computations with the aid of standard mathematical tables. However, in the event that extensive use of a model was envisioned, or if the model was to be applied to the representation of a particularly large real-world system, then it was recommended that tabular solutions be generated with the aid of a digital computer. The ease of implementing solutions on a digital computer was discussed in the concluding chapter of the thesis.

In recognition of the newness of the research area and in an attempt to provoke and encourage future investigations, both qualitative and quantitative portions of the thesis emphasized the identification of potentially useful methods for formulation and solution of model equations. The general usefulness of formulation schemes and solution methods applied to the derivation of repairman models with spares was discussed in some detail in terms of results obtained and obtainable through extensions. In addition, a large number of potentially useful, but untested, methods were described in the extensive (250-reference) literature survey.

It is thought that the research results will be of particular interest to the fleet-type transportation and maintenance service industries. However, in general, the research results are expected to have widespread application to problems involving the operation, maintenance, cost, reliability, and availability of systems comprised of groups of similar components.

CHAPTER I

INTRODUCTION

Purpose

The purpose of this research is to extend--and to provide for the future extension of--the bounds of knowledge relating to the design of maintenance facilities and spares-stocking policies for systems which are essentially characterized by the use of rotatable, fleet-operated components. Analytic aspects of the investigation will emphasize the modeling of congestion phenomena associated with the flow of nonoperative components through a limited-capacity service facility and the subsequent storage of these components before returning them to use.

It is thought that the research results will be of particular interest to the fleet-type transportation and maintenance service industries. However, in general, the research results are expected to have widespread application to problems involving the operation, maintenance, cost, reliability, and availability of systems comprised of groups of similar components.

The Systems of Interest and Their Representation in the Present Investigation

Characteristics

A fleet-type transportation system is essentially characterized by the use of vehicles or other devices to move passengers and/or goods from one location to another. Such systems are found in association with nearly every aspect of modern human existence and endeavor: airlines, railroads, ship lines, trucking firms, subways, taxicab companies, milk delivery systems, forklift truck fleets, etc. The interchangeable, fleet-operated components of such systems may be regarded as being the vehicles themselves (e.g., airplanes, locomotives, trucks) or some smaller component used on the vehicles (e.g., transmissions, engines, frames, appliances).

In all such transportation systems, the maintenance subsystem associated with a particular fleet-operated component exists to fulfill certain requirements--repair, renewal/replacement, or both--dictated by the failure and service characteristics of that component and, to a lesser extent, by the physical layout of the transportation system. Of course, the same maintenance facilities may be used to service more than one type of component. In such cases, the maintenance subsystem associated with a particular component may be regarded as having ancillary functions which limit and otherwise define its availability and capacity with respect to the given component.

Representation in the Present Investigation

In the present investigation, the systems of interest will be represented in parametric form, that is, via a specification in general symbolic terms of such distinguishing characteristics as (1) number of components comprising the system, (2) average duration of operating time of a typical component, (3) average time required to service a typical component, (4) number of service channels available for component repair, (5) number of spare components available, and (6) rules defining the order of service.

The variable operating conditions encountered in such systems will be represented (stochastically) by a statement of the range and frequency of occurrence of possible operation and service times for a typical component. These frequency distributions will themselves be parametrically general in the sense that they will be specified in terms of one or more symbolic factors.

An Example

An example of particular interest to the present investigation is the system in which aircraft engines are maintained by a commercial airline. One of the first studies of a spares-provisioning problem was made in 1954 by British Overseas Airways Corporation analysts, J. Taylor and R. R. P. Jackson [232], who depicted the engine maintenance cycle as being of the form shown in Figure 1.

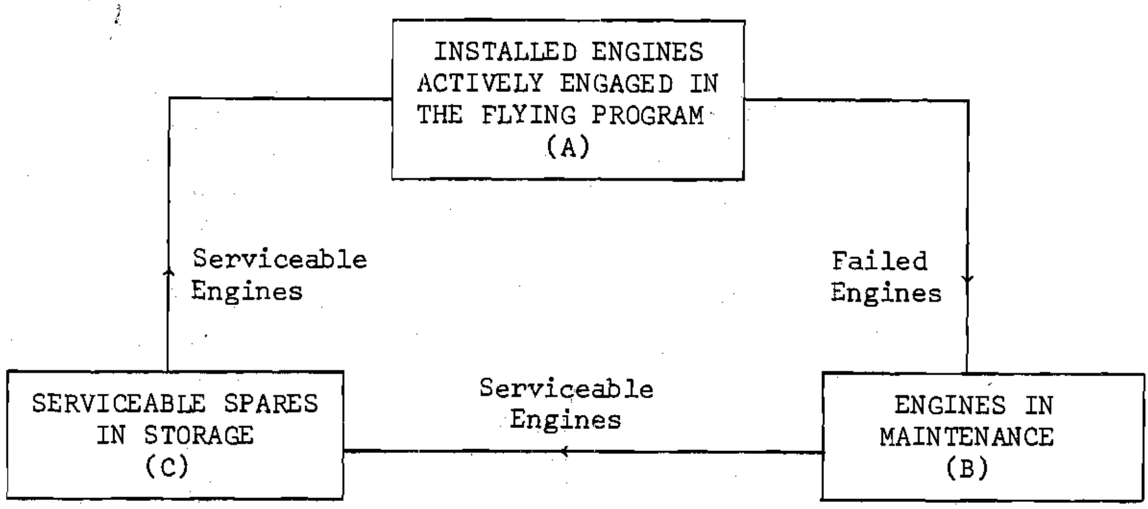

Figure 1. The Commercial Airlines' "Engine Pipeline"

The parametrically-general model which Taylor and Jackson devised to represent this system is described in detail in Chapter III. For the moment, it is only observed that their model is quite similar in formulation to the classical "repairman model" studied by Palm [173] and others (see Chapter II), and that this similarity suggests that it be called a "repairman model with spares."

Model Nomenclature

The flow system of general interest in the present investigation is that illustrated in Figure 1. The research results are intended to apply to any system which has a group of interchangeable components operated in this manner. For historical reasons, the models that will be devised to represent flow and congestion phenomena associated with such systems will be referred to as "repairman models with spares." Similarly, the underlying systems will be called "repairman systems with spares."

Limitations of the Literature

Only a few examples of repairman models with spares have appeared in the literature. One of these was Taylor and Jackson's model [232] of aircraft engine maintenance; another was Toft and Boothroyd's model [240] devised to represent colliery operations; and a third [244] was an application of Taylor and Jackson's model to represent maintenance activities for an analogue computer. This literature is described in detail in Chapter III.

Importance of the Topic

In a general sense, the importance of the research topic is demonstrated by the prospective widespread applicability of the research results. The profit realizable from operation of a fleet-type transportation system is directly influenced by the costs of vehicle ownership and maintenance. Also, the competitive success of these systems is strongly influenced by the consistency with which they meet fixed operating schedules, and this consistency depends directly upon the contribution of maintenance support to system availability. With respect to other examples of potential application, it is observed that operating cost and system availability are almost invariably the factors upon which a system depends for its functional, economic, and competitive success.

The ability to design maintenance facilities and spares-stocking policies to optimize operating cost and system availability clearly depends upon an intimate knowledge of system behavior. The qualitative and empirical approaches that have dominated previous studies of fleet-type system behavior, while valuable as applied to specific real-world systems, have limited flexibility as general investigative and planning tools. On the other hand, parametrically general models--for example, of the repairman-with-spares type sought in the present investigation--have the advantage of providing a relatively inexpensive, easily manipulated, and generally applicable planning tool for comparisons of variations on existing systems and for the study of proposed systems.

Objectives and Scope of Study

Purpose and Objectives

As previously stated, the purpose of this research is to extend--and to provide for the future extension of--the bounds of knowledge relating to the design of maintenance facilities and spares-stocking policies for systems which are essentially characterized by the use of rotatable, fleet-operated components.

The considerations related in this chapter motivate the choice of research objectives. The general objective of the investigation is the development of a theoretical basis for the representation and study of congestion phenomena associated with the flow of nonoperative components through a limited-capacity service facility and the subsequent storage of these components before returning them to use. A number of specific objectives are identified with attainment of this general objective. The primary specific objectives involve the development of models to represent the systems of interest.

1. Development of "repairman models with spares;" that is, development of queueing models of the classical repairman type, but which include the additional ramification of spares provisioning.

2. Development of variations on the existing models which may be identified as "repairman models with spares," in order to increase their potential usefulness.*

The secondary specific objectives recognize the limitations of

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*The existing repairman models with spares are Taylor and Jackson's model [232] of an aircraft-engine maintenance system and Toft and Boothroyd's model [240] of colliery operations. These models are described in Chapter III.

the literature on repairman models with spares in providing a source of theory, analytic techniques, and procedural philosophy of potential usefulness to the present and future investigations:

3. Presentation in one location of a comprehensive survey of the literature on repairman models, the direct historical and natural methodological precursors to repairman models with spares.

4. Contrast of the relative bounds of knowledge on regular repairman models and those with spares in order to give (a) a measure of the relative level of development of theory on repairman models with spares and (b) an indication of the unsolved repairman problems with spares for which solutions would be desirable and worthwhile.

5. Demonstration of the relationships among repairman models with spares, repairman models, cyclic and finite queueing models, and networks of queues; and evaluation of their implications on the study of repairman models with spares.

Study Procedure

General Outline. The quantitative aspects of the investigation (objectives 1-2) are viewed as being essentially the beginnings of a new area of research within the general realm of queueing theory applications. Accordingly, it was felt necessary to attain the secondary objectives 3-5 before attempting to fulfill the primary objectives 1-2. Chapter II provides a discussion of the characteristics of repairman models and a survey of the published work on repairman systems (objective 3). Chapter III extends the discussion to the limited literature on repairman models with spares (objective 4) and introduces additional areas of the queueing theory literature that are pertinent to the study of repairman models with spares (objective 5). The foundation of methods and procedural philosophy collected in Chapters II and III is applied to the development of new repairman models with spares in Chapters IV and V (objectives 1-2).

Model Development. The analytic procedures which will be used in the derivation of new repairman models with spares are quite varied in view of the number of different problems which will be considered in Chapters IV and V. The principal methods of attack will be those of Poisson queueing theory (see aspects outlined at end of Chapter I), those of cyclic and finite queueing network theory (see survey in Chapter III), and a "method of stages" for the decomposition of chi-square distributed phenomena (see Chapter V).

Form of Results

In most cases, the model developed will be a threefold representation of a system of interest:

(1) A set of differential-difference equations relating the time-dependent probabilities $P_n(t)$ of there being n units requiring service at time t; for example, equations relating $(d/dt)P_n(t)$ to $P_{n-1}(t)$, $P_n(t)$, $P_{n+1}(t)$, etc. This will be referred to as the "time-dependent" or, equivalently, "dynamic" model of the system.

(2) A set of finite difference equations relating the proportions of time p_n during which n units require service when the system has reached "statistical equilibrium" after operating for a long time ($P_n(t) \to p_n$ as $t \to \infty$). This will be referred to as the "steady-state" or,

equivalently, "stationary" model of the system.*

(3) A set of formulae explicitly defining the p_n in terms of the system parameters. The formulae will usually result from analytic solution of the stationary equations, although in some cases additional background material will be used to shorten the analysis. These formulae also comprise a stationary model of the system.

In one instance, explicit formulae will be developed for the $P_n(t)$; namely, an explicit time-dependent representation will be developed for the same repairman system with spares that Taylor and Jackson [232] studied in the steady-state.

All results, explicit or implicit, will be expressed parametrically in terms of (1) number of interchangeable components in the system, (2) average duration of operating time of a typical component, (3) average time required to service a typical component, (4) number of service channels available for component repair, (5) number of spare components available, and, in some cases, (6) an extra parameter included to generalize the service-time distribution.

Scope and Limitations

The scope of the investigation is delimitable in terms of the variety of models to be developed. Chapter IV will present a number of models, results, and observations for repairman systems in which the units have negative-exponentially distributed failure and service times.

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*Stationarity is in fact attained for any value of t for which $(d/dt)P_n(t) = 0$. In the models of interest, it is expected that this condition will prevail for large t.

Topics to be treated are: spares which fail in storage, artificially extended service intervals, reduction of system failures, the prospects for time-dependent solutions, repairmen with ancilliary duties, faulty repairs, cyclic queues, transient delays, and the servicing of aircraft engines. One or more repairman models with spares will be introduced in association with each topic. Chapter V will present several repairman models with spares in which failure times follow the two-parameter chi-square distribution with even degrees of freedom.

Abbreviated Queue Designators

D. G. Kendall [129] has devised a concise notational scheme for the identification of queueing models. This scheme will be adopted in the present discussion. In Kendall's popular notation, $M|G|c \equiv \{$type M interarrival-time density function|type G service-time density function| c servers$\}$; where, by convention, it is assumed, unless otherwise qualified or clarified, that servicing is in a simple, non-preferential order (e.g., first-in-first-out) and that an infinite number of waiting positions is provided. Kendall's system uses M to denote a Poisson number-of-occurrences distribution or, equivalently, a negative-exponential interoccurrence-interval distribution; D for a deterministic distribution (e.g., constant interoccurrence intervals); E_k for the *kth* Erlang distribution; G for a general (arbitrary) distribution; and GI for a general distribution with independence assumptions.

Some Aspects of Poisson Queueing Theory

It is assumed that the reader has some familiarity with basic Poisson queueing theory. This section will review only a few concepts and results that are of general and recurring significance to the development of succeeding chapters. Reference is made to Feller [52, 53], Karlin [116], and Syski [212] for a more comprehensive theoretical discussion of the same material and to Saaty ([196], Chapters 4-5) for additional examples of its application. Additional aspects of Poisson queues will be described later in the context of the literature survey and the presentation of results of the current investigation.

Particular attention should be paid to the description of the general birth and death process. The method of formulation and solution outlined for it may be regarded as a prototype for much of the analysis in the present investigation. The concluding paragraphs of this section explain how the precepts of its construction will be extended to the development of models with multi-dimensional state changes in Chapters IV and V.

The Poisson Process

The Poisson distribution is frequently used to represent input phenomena (called "Poisson input") where arrivals occur essentially "at random" or when little or nothing is actually known about the input process. The basis for such usage stems from several causes: First, the Poisson distribution may be derived from very general assumptions (given below) which agree quite well with our notion of what the probabilistic properties of a random phenomenon might be. Second, the ease with which computations may be effected overrides many possible objections.* Third, and most important, many real input systems (e.g., road traffic, telephone calls, machine breakdowns, restaurant customers, etc.) show excellent empirical correspondence to their Poisson analogues.

Postulates for the Poisson process are: Whatever the number of arrivals during the interval of time $(0,t)$, the probability that during the small interval $(t,t+\Delta t)$ an arrival occurs is $\lambda\Delta t + O(\Delta t)$, where λ is a real constant; and the probability that more than one arrival occurs is $O(\Delta t)$.**

Let $\Phi_a(t)$ denote the probability that precisely a arrivals occur during $(0,t)$. Then, it can be shown that, for the Poisson process,***

$$\Phi_a(t) = \frac{(\lambda t)^a}{a!} e^{-\lambda t}, \quad t\geq 0, \quad a=0,1,2,\cdots, \tag{1}$$

which is known as the Poisson distribution. The expected number of arrivals during $(0,t)$ is λt and, accordingly, the mean arrival rate is λ.

Further, it can be shown that successive interarrival intervals are independent and that the time t between any two successive arrivals (or the waiting time to the first arrival) has the negative exponential

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*E. C. Molina [157] has published extensive tables of individual and complementary-cumulative terms of the Poisson distribution.

**$O(x)$ is used to denote a quantity of smaller order of magnitude than x. More precisely, $O(x)$ stands for a quantity such that $x^{-1}O(x)\to 0$ as $x\to 0$.

***The Poisson process is an example of a pure birth process. A set of differential-difference equations in $\Phi_a(t)$ can be formulated and solved in the manner described later for the general birth and death process.

distribution,

$$g(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \tag{2}$$

with mean $1/\lambda$. This equation is often used to represent the service-time distribution for a unit in queueing problems.

Markovian Property of the Negative Exponential Distribution

In general, we say that a process is Markovian in nature if the probability of some future event is dependent, at most, upon the present state of the system. That is, a Markovian process has no "memory" of its previous behavior. The negative exponential distribution is Markovian, for

$$\Pr\{T>t\} = \int_t^\infty \lambda e^{-\lambda t}\, dt = e^{-\lambda t} \tag{3}$$

and

$$\begin{aligned}
\Pr\{T>t+x \mid T>x\} &= \frac{\Pr\{T>t+x\}}{\Pr\{T>x\}} \\
&= \frac{e^{-\lambda(t+x)}}{e^{-\lambda x}} \\
&= e^{-\lambda t} \\
&= \Pr\{T>t\}.
\end{aligned} \tag{4}$$

The General Birth and Death Process*

Several sections of Chapters II, III, and IV make use of properties of the general birth and death process. These will be described in the context of queueing phenomena in which births are equated to arrivals, deaths to service completions, and the population to the number of units in the system (i.e., the total number of units either being serviced or awaiting service).

Notation. For the state where there are n ($n=0,1,2,\cdots$) units in the system, let S_n denote the state of the system; λ_n, mean rate at which units arrive in the system; and μ_n, mean rate of serving units. Let $P_n(t)$ be the probability that the system is in state S_n at time t and let p_n be the corresponding steady-state probability of being in state S_n.

Postulates. The system changes only through transitions from states to their nearest neighbors ($S_n \to S_{n-1}$ or S_{n+1} if $n \geq 1$ but $S_o \to S_1$ only). If at any time t the system is in state S_n, the probability that during the small interval $(t, t+\Delta t)$ the transition $S_n \to S_{n+1}$ occurs is $\lambda_n \Delta t + O(\Delta t)$, and the probability of $S_n \to S_{n-1}$ ($n \geq 1$) is $\mu_n \Delta t + O(\Delta t)$. The probability that two or more transitions occur is $O(\Delta t)$.

Time-Dependent Model. The postulates provide a means for relating the state of the system at time $t + \Delta t$ to its state at time t through the consideration of a set of mutually-exclusive possible

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*The general birth and death process was developed by William Feller during 1936-1940. The current discussion follows his exposition ([53], Chapter 17), with stationary solution taken from Karlin [116], p. 194.

happenings during the small interval Δt. The probabilities of these events are thus additive and the following equations can be written:

$$\left.\begin{aligned} P_o(t+\Delta t) &= (1-\lambda_o\Delta t)P_o(t) + (1-\lambda_1\Delta t)(\mu_1\Delta t)P_1(t) + O(\Delta t); \\ P_n(t+\Delta t) &= (1-\lambda_n\Delta t)(1-\mu_n\Delta t)P_n(t) + (\lambda_{n-1}\Delta t)(1-\mu_{n-1}\Delta t)P_{n-1}(t) \\ &\quad + (1-\lambda_{n+1}\Delta t)(\mu_{n+1}\Delta t)P_{n+1}(t) + O(\Delta t), \\ &\qquad n=1,2,3,\cdots. \end{aligned}\right\} \quad (5)$$

Subtracting $P_n(t)$ from both sides, dividing by Δt, and taking the limit at $\Delta t\to 0$, we obtain the dynamic equations of state,

$$\left.\begin{aligned} \frac{dP_o(t)}{dt} &= -\lambda_o P_o(t) + \mu_1 P_1(t); \\ \frac{dP_n(t)}{dt} &= -(\lambda_n+\mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \quad n=1,2,3,\cdots; \end{aligned}\right\} \quad (6)$$

where, if the system started in state S_k at $t=0$, the initial conditions are

$$P_k(0) = 1 \quad \text{and} \quad P_n(0) = 0, \quad n\neq k. \qquad (7)$$

It is known that, for arbitrary prescribed coefficients $\lambda_n \geq 0$, $u_n \geq 0$ there always exists a positive solution $\{P_n(t)\}$ such that $\sum_n P_n(t) \leq 1$. Further, if the sequences $\{\lambda_n\}$, $\{\mu_n\}$ are bounded (or increase sufficiently slowly), this solution is unique and satisfies the regularity condition,*

$$\sum_{n=0}^{\infty} P_n(t) = 1. \tag{8}$$

Steady-State Model. It can be shown (either from the explicit form of the solutions or from the general ergodic theorems for Markov processes) that under any circumstances the limits, $\lim_{t\to\infty} P_n(t) = p_n$, exist; they are independent of the initial conditions (7); and $\lim_{t\to\infty} (d/dt)P_n(t) = 0$. The steady-state equations which are obtained from (6) as $t\to\infty$ are

$$\left.\begin{aligned} 0 &= -\lambda_o p_o + \mu_1 p_1; \\ 0 &= -(\lambda_n+\mu_n)p_n + \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1}, \quad n=1,2,3,\cdots. \end{aligned}\right\} \tag{9}$$

Stationary Solution. Equations (9) may be evaluated recursively to obtain expressions for $p_1/p_o, p_2/p_o, p_3/p_o, \cdots$. A general expression for p_n/p_o then follows by induction. For the case where the λ_n are finite non-negative, the μ_n are finite positive, and the sequence $\{p_n/p_o\}$ convergent, a unique solution may be written in the form**

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*For unique solutions to repairman models (with or without spares), it is required only that the λ_n, μ_n be finite non-negative. Then, boundedness of $\{\lambda_n\}, \{\mu_n\}$ is assured since there are only a finite number of states (e.g., $n=0,1,2,\cdots,N$ for some prescribed integer N).

**Equations (10-11) are correct, but restricted to the range of definition when the state space is bounded (e.g., to $n=0,1,2,\cdots,N$) as

$$p_n = \frac{\lambda_o\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_n} p_o, \quad n=1,2,3,\cdots, \tag{10}$$

where p_o is to be determined from the normalizing condition,

$$1 = \sum_{n=0}^{\infty} p_n. \tag{11}$$

Solutions for cases where these conditions do not hold are quite irregular in that they either reduce to a trivial result or produce an obvious variation on the main solution. Such irregular cases have rarely proved to be of much practical significance in the past and, accordingly, are generally excluded from consideration in all models developed in the present investigation.

Poisson Input. Feller has shown that processes with negative-exponentially distributed transition times are the (only) ones fitting this birth and death model.* Due to the equivalence between a Poisson frequency-of-occurrence distribution and a negative exponential *inter*occurrence-time distribution, the input process of the general birth and death system is sometimes referred to as "state-dependent Poisson input"; however, it should be noted that the process is not true Poisson input unless $\lambda_n \equiv \lambda$, a constant, for $n=0,1,2,\cdots$. The Poisson postulates exclude any notion of state dependency.

Applications. Equations (6-11) provide a means for easily

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for the repairman models and repairman models with spares treated in later chapters.

*Proof is started in the cited reference ([52], Chapter 17, § 3) and concluded in Volume II of his book ([53], pp. 8-9).

obtaining models for many queueing situations of interest. For example, the M|M|c model is given by $\lambda_n \equiv \lambda$; $\mu_n = n\mu$, $n=1,2,3,\cdots,c$; $\mu_n = c\mu$, $n=c,c+1,c+2,\cdots$; $\lambda<c\mu$. Palm's repairman model [173] was shown to be an example in Feller's exposition on the general process ([52], § 17.7). Further, as will be shown in Chapter III, the repairman models with spares laboriously derived by Taylor and Jackson [232] and Toft and Boothroyd [240] could have been more simply constructed had the authors merely specified an appropriate set of λ_n and μ_n for the general birth and death equations. Indeed, many of the queueing models referenced in Chapters II and III follow from the general birth and death model. Unfortunately, existence of this general model seems to have been often overlooked in the literature.

Formulations for More Complex Processes

The general birth and death process described above is an example of continuous-time Markov processes with discrete state space and time-homogenous transition probabilities. Its distinguishing characteristic is the confinement of transitions to nearest neighbors. In Chapters IV and V, we will have occasion to relax this restriction and consider systems in which transitions are permitted between non-neighboring states. The emphasis in Chapters IV and V will be on the formulation and (steady-state) solution of model equations. Rather than entering into a discussion of postulates and solution-existence criteria for each new model, we will treat these topics once here.

Postulates. The revised postulates are completely analogous to those for the birth and death process. They are: If at any time t the system is in state S_i, the probability that during the small interval

$(t,t+\Delta t)$ the transition $S_i \to S_j$ occurs is $\lambda_{ij}\Delta t + O(\Delta t)$, where λ_{ij} is a real constant. The probability that two or more transitions occur is $O(\Delta t)$.

Equations of State. The various equations of the system are obtained from a development similar to that for the general birth and death model. For a finite state space, say, $\{S_n,\ n=0,1,2,\cdots,N\}$--which is all that will be required for the study of repairman models with spares--these equations are

$$P_n(t+\Delta t) = \left[1 - \sum_{\substack{k=0\\k\neq n}}^{N} \lambda_{nk}\right] P_n(t) + \sum_{\substack{k=0\\k\neq n}}^{N} \lambda_{kn}P_k(t) + O(\Delta t), \qquad (12)$$

$$n=0,1,2,\cdots,N;$$

$$\frac{dP_n(t)}{dt} = -\left[\sum_{\substack{k=0\\k\neq n}}^{N} \lambda_{nk}\right] P_n(t) + \sum_{\substack{k=0\\k\neq n}}^{N} \lambda_{kn}P_k(t), \quad n=0,1,2,\cdots,N; \qquad (13)$$

$$0 = -\left[\sum_{\substack{k=0\\k\neq n}}^{N} \lambda_{nk}\right] p_n + \sum_{\substack{k=0\\k\neq n}}^{N} \lambda_{kn}p_k, \quad n=0,1,2,\cdots,N; \qquad (14)$$

with initial and normalizing conditions given by Equations (7-8,11) when these conditions are restricted to the finite range of definition.

Conditions for Application.* Finiteness and non-negativity of

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*The remarks of this section apply equally to the general birth and death process having a finite number of states since it is a special case of the process described here. See Syski ([212], pp. 210-212) for a treatment of the general birth and death process in this specialized form.

the λ_{ij} (i.e., $0 \le \lambda_{ij} < \infty$) are sufficient conditions for:

(1) the postulates to define a unique, conservative Markov process,* with a finite Markov chain embedded at state transition epochs and (only) negative-exponentially distributed holding times in a state (see [52], § 17.3, [53], pp. 8-9; [116], pp. 226-227);

(2) the existence of $(d/dt)P_n(t)$ and continuous $P_n(t)$, $0 \le t < \infty$ (see [116], § 7.8);

(3) the existence of a unique positive solution $\{P_n(t)\}$ satisfying Equations (13,7-8) (see [116], § 7.8; [212], pp. 120-121); and

(4) the existence of a unique positive steady-state distribution $\{p_n\}$ satisfying Equations (14,11) independent of the initial conditions (see [53], § 14.9; [212], p. 121).

Solution of Equations. Matrix notation can be used to write, at least formally, solutions for the $P_n(t)$ and p_n (see [116], § 7.8). However, these general, indicated solutions are difficult to implement since one must still perform a number of non-trivial operations on the (general) *Nth* order matrix of coefficients $[\lambda_{ij}]$. In cases where the form of the λ_{ij} is known (e.g., $\lambda_{ij}=i\lambda$) or some λ_{ij} equal zero, it often proves more practical to obtain recursive solutions directly from the state equations (cf. the birth and death equations). This second approach is used consistently in the present investigation.

*Actually, here, as for the general birth and death process, we refer to the existence of a unique "minimal" process (see [116], § 8.3 and references cited therein).

CHAPTER II

A SURVEY OF THE LITERATURE ON REPAIRMAN MODELS

Introduction

This chapter will present a review of the significant literature on repairman models as background to the survey of the literature on repairman models with spares which follows in Chapter III. The relevance of the literature on repairman models to the study of repairman models with spares is multifold. First, repairman models are the direct historical and methodological precursors to repairman models with spares. Accordingly, the literature on repairman models provides a rich source of theory, analytic techniques, and procedural philosophy of potential usefulness to any study in the realm of repairman models with spares. As such a source, this body of literature is essential since the comparatively sparse volume of material on repairman models with spares cannot provide such a background (cf., Chapter III).

Second, it may be observed that repairman models are a special case of repairman models with spares; that is, the case where the spares parameter is zero. Thus, the knowledge of solutions for regular repairman models can serve as a partial check on results obtained for analgous repairman models with spares. Third, as will be elucidated in Chapter III, both regular repairman models and those with spares can be regarded as special cases of cyclic queueing models.

Finally, a contrast of the relative bounds of knowledge on regular repairman models and those with spares gives a measure of the current level of development of theory on the repairman models with spares. One may also derive from such a comparison some notion of the unsolved repairman problems with spares for which solutions will be of interest and value to other researchers and users in the field. Accordingly, an attempt has been made to anticipate the needs of future researchers in the field by giving a broad scope to this survey of the literature on repairman models.

A common assumption in the vast majority of the papers reviewed is that steady-state conditions prevail. The reader may find it helpful to regard this assumption as a convention; all exceptions will be clearly identified.

Particular attention should be paid to the discussion of the models of Palm [173] and Takács [219]. These will play an important role in the later discussion of repairman models with spares provisioning.

Characteristics of Repairman Models

"Repairman models" or, equivalently, "machine interference models" belong to that portion of queueing theory* which treats the finite-source arrival process. The relationship between the general queueing model (or "queue") and the repairman model is illustrated in Figures 2 and 3. In all repairman models, the input is generated according to some rule from a finite pool of potential arrivals, usually

*A few of the models discussed are obtained from approaches other than queueing theory.

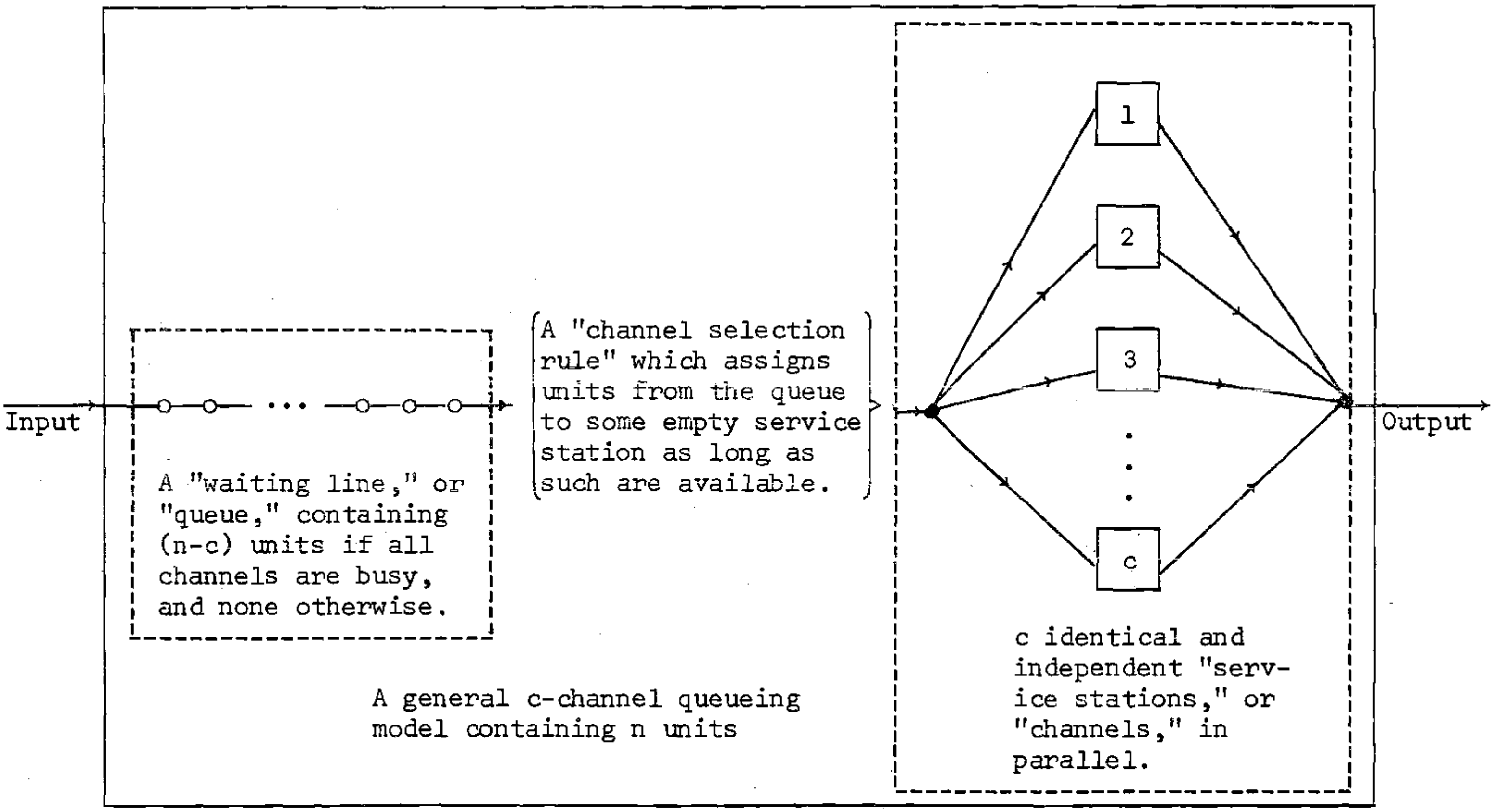

Figure 2. The General Multichannel Queueing System

identified as machines which break down from time to time and require service before being returned to duty. After receiving service, a departing unit returns to the pool as shown in Figure 3. In most repairman models, the input process is assumed to be proportionately dependent upon the number of units currently in the pool.

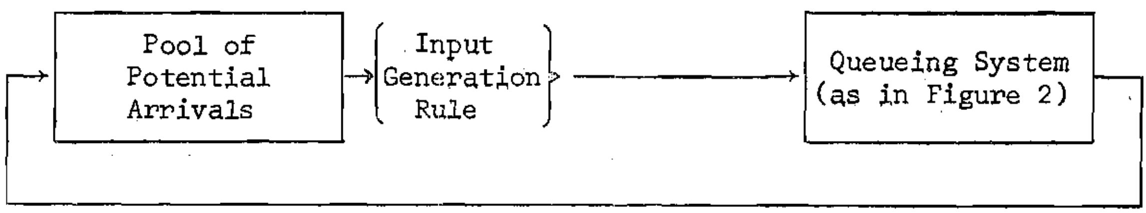

Figure 3. The Repairman Model

Proportional dependence occurs under the assumptions that (1) all units are identical in their essential failure characteristics and (2) the time to failure of an operating unit (one in the pool) depends only upon the elapsed time since its last service completion and *not* upon the status of other units in the system. A case that has been of particular interest is that where the time to failure of a unit is a negative exponential variate with parameter λ. It follows in this event that, if k units are observed to be operating at some time, the waiting time to the first failure occurrence (of any unit) is negative-exponentially distributed with parameter $k\lambda$, irrespective of the times at which each of the k units was actually placed in service.[*] The input process for this

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[*]The argument is based on (1) the Markovian ("forgetfulness") property of the negative exponential distribution (see Chapter I) and (2) independence of the unit failure processes. Note that

case is *not* Poisson since it violates the Poisson assumption of state-independency; however, a natural, if somewhat misleading, usage in the literature has been to refer to it as "Poisson input from a finite source."

The "solution" to a repairman problem is usually thought of as being specification of the probability distribution of the number of failed units.

Origin of the Repairman Model**

The precise origin of repairman models is difficult to place because (1) the underlying analytic techniques--especially queueing theory--were well-developed in other areas of research--especially telephony--before being applied to repairman problems;*** (2) many of

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$$\Pr\{T>t\} = \Pr\{T_1>t,T_2>t,T_3>t,\cdots,T_k>t\} = \prod_{j=1}^{k} \Pr\{T_j>t\}$$
$$= \prod_{j=1}^{k} e^{-\lambda t} = e^{-k\lambda t},$$

where T denotes the waiting-time variate, the T_j are the unit failure-time variates, and the form of the complementary-cumulative negative-exponential distribution is taken from Equation (3).

**Many of the references cited in this section are not readily accessible. An effort has been made to indicate, parenthetically or via footnote, the existence of more accessible secondary material which relates the substance of these early papers.

***The ideas underlying machine interference theory are mathematically analogous to those associated with telephony for the case where a call is lost if all lines are busy. This analogy is discussed by Jackson and Adelson ([100], pp. 19-20), Cohen ([31], pp. 138-139, 142-143), and to a limited extent by Saaty ([196], §§ 14.2,14.6,14.7). The analogy has received considerable attention in Takács work ([214-228], *passim)*. Cohen [31] gives a particularly illuminating example comparing the landmark papers of Erland [49,50], on queueing theory applied to telephony, and of Palm [173], on queueing theory applied to machine the early papers appeared in obscure sources and were not brought to the attention of later researchers who (independently) duplicated this earlier work; and (3) there were many early papers which (inadequately) treated the problem in a semi-conjectural and unsophisticated manner.*

Credit for the first significant mathematical formulation (but not a solution) of a repairman problem should be assigned to A. K. Erlang [49,50], 1909 and 1917.** In his 1909 paper [49] on telephone traffic with a single server, Erlang initially considered the mathematical implications of an amended "Poisson input" generated by a finite number of subscribers. However, in developing his model, Erlang found it expedient to drop consideration of the number of occupied independent callers so that his final solution in fact reflected the case of an infinite number of subscribers and true Poisson input. His 1917 paper [50] extended the analysis to the case of several servers. He assumed either negative-exponential or constant holding times ([196], pp.20-

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interference. However, Cohen's statement "that both problems are mathematically identical" is misleading and should be interpreted in the light of remarks in § (g) of his article. Riordan's discussions ([189], *passim*) of delay and loss systems may also prove of interest.

*This is not to say that such efforts were trivial or insignificant at the time of publication. They were representative of the then existing state of the art and probably did much to stimulate later, more adequate treatment of repairman problems.

**These two works are considered by many to be the founding papers of queueing theory (e.g., [52], p.412; [196], p.21; [40], p. 160; [31], p.138). Brockmeyer, Halstrøm and Jensen [18] have published English translations of most of Erlang's papers in a memoir which also contains Jensen's review (pp.23-100, "An Elucidation of Erlang's Statistical Works through the Theory of Stochastic Processes") of Erlang's work from a modern viewpoint.

21).* Erlang's papers are especially noteworthy since essentially the same approach was used by Palm [173] some 30 years later to solve (as a machine interference problem) for the case of a finite number of subscribers.

The first mathematical discussion of machine interference, *per se*, is usually attributed to Khinchin** [133], 1933. Khinchin formulated, but did not solve explicitly, a machine interference problem involving a finite population, negative-exponentially distributed machine failure times, arbitrarily (general independent) distributed service times, and a single repairman ([13], p.66; [164], p.106; [196], p.323). Kronig [142] and Kronig and Mondria [143], 1943, independently studied the same problem and obtained an explicit expression for the expected waiting time before service of a failed machine.*** More extensive results were obtained for this problem by Palm [173] and Ashcroft [3] during the period 1947-1950. The case of constant service times was investigated by Westgarth [247], Ashcroft [3], and Benson and Cox [13] during the period 1948-1951.

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*Erlang's loss formula is valid for arbitrary service time distributions; his delay formula is valid only for negative exponential service times.

**An alternative, often-encountered transliteration (from Russian) of the author's name is "Khintchine."

***According to Takács [219], the papers of Khinchin [133], Kronig [142], and Kronig and Mondria [143] all make use of an erroneous assumption which leads to an improper generalization of certain results. As a consequence, the given expressions for a failed machine's waiting time for service are not generally valid, but apply only in the case of negative-exponentially distributed service times.

Prior to and concurrent with this early theoretical work, a number of researchers--Bernstein [16], Duvall [250], Field [55], Frantz [59], Gautzsch [65], Gross [73], Jones [109-111], Rodhe [190], Stout [208], Walz [245], Weir [246], Wright [250], and others--advanced approximate solutions or empirical formulae for problems of machine interference.* The paper by Wright [250] is probably best known. It was essentially a modification of a solution by Fry [60] for a similar problem in the availability of telephone lines. It assumed that failures were distributed at random in time, independent of the number of machines operating (Poisson input),** and that each failure required the same servicing time (constant service time distribution). The resulting formula for (approximating) interference was found to be empirically valid for cases where an operator had charge of six or more machines.

A somewhat singular approach in the early work on machine interference was taken by W. D. Jones [110,111], 1949. He developed a machine-interference computer which permitted simulation of two to ten machines tended in rotation by one operator. The computer contained a timing and revolution counting device which provided the basic data

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*Surprisingly, in view of (1) the many excellent, well-founded models, charts, and tables which have been developed in formal machine interference theory and (2) the now widespread availability of desk calculators and digital computers; numerous "approximation" formulae continue to be published each year, principally in the various trade journals of the automated production industries. Here reference is made, *not* to the natural precursors of formal theory such as the papers cited above and elsewhere in this chapter, but to the seemingly pointless proliferation of approximate solutions to already "well-solved" problems.

**An equivalent assumption is that the failure rate of a particular operating machine is inversely proportional to the number of machines operating!

necessary for computation of the servicing, running, and interference times of the machines for each test.

Devoted Repairmen, the Era Following Palm

Characteristics of Devoted Repairman Models

Problems involving devoted repairmen form the mainstream of machine interference literature. Early work in this area was described in the preceding section. The present section will discuss work which, for the most part, proceeds from the analytic methods developed by Palm [173].

The term "devoted repairman" is used here to denote a repairman who has no duties beyond the attendance of his assigned machines and who responds immediately whenever a machine fails, unless preoccupied with a previously failed machine. It should be noted that this definition includes the case of devoted, real-world repairmen who in fact have ancillary (non-repair oriented) duties, but where these duties are performed only during idle periods when all machines are operative and/or these duties are closely associated with the actual repair function. In the mathematical sense, it will be said that a repairman is "devoted" when his duties may be completely allocated among the various machines; that is, when a single parameter--the service time for a machine--can be used to describe the repairman's non-idle period activities.

Palm's Model

The methodological basis underlying much of modern machine interference theory was provided by Palm [173] in his three 1947 papers.*

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*Actually, Palm's papers (written in Swedish) did not receive

The assumptions of Palm's model were:

1. There are c repairmen who are similar as to service skill and aptitude; there are N ($N \geq c$) machines which are similar as to average breakdown incidence in unit working time and as to skill required in servicing them.

2. Uninterrupted working time for a machine and service time on a machine are both negative-exponentially distributed random variables with parameters λ and μ, respectively.

3. All random variables are independently distributed.

4. The system is in a state of statistical equilibrium.

Let S_n denote the state of the system; λ_n, the (combined) rate at which machines are failing; and μ_n, the (combined) rate at which machines are being restored to service when there are n ($n=0,1,2,\cdots,N$) nonoperative machines in the system. Let p_n denote the stationary probability that the system is in state S_n.

Then, the steady-state equations derived by Palm can be stated as

$$\left.\begin{array}{ll} N\lambda p_o = \mu p_1; \quad c\mu p_N = \lambda p_{N-1}; & \\ [(N-n)\lambda+n\mu]p_n = (N-n+1)\lambda p_{n-1} + (n+1)\mu p_{n+1}, & n=1,2,3,\cdots,c-1; \\ [(N-n)\lambda+c\mu]p_n = (N-n+1)\lambda p_{n-1} + c\mu p_{n+1}, & n=c,c+1,c+2,\cdots,N-1. \end{array}\right\} \quad (15)$$

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widespread attention until 1950 when a detailed summary (in English) appeared in Feller's popular book ([52], pp.379-383, First Edition; pp.416-421, Second Edition; pp.462-466, Third Edition). An English translation [174] was published in 1958; however, it seems to have escaped general attention.

It can be seen that these equations are equivalent to those of the general birth-and-death model (9) for the case

$$\left.\begin{array}{lll} \lambda_n = (N-n)\lambda, & \mu_n = n\mu, & n=0,1,2,\cdots,c; \\ \lambda_n = (N-n)\lambda, & \mu_n = c\mu, & n=c,c+1,c+2,\cdots,N; \end{array}\right\} \quad (16)$$

and $\lambda_n = \mu_n \equiv 0$ for all $n>N$. Indeed, substitution of Equations (16) into Equations (6,10-11) will yield the dynamic equations of state and the stationary solution to Palm's problem.*

Palm solved Equations (15) recursively to obtain

$$\left.\begin{array}{ll} p_n/p_o = \left(\frac{N\lambda}{\mu}\right)^n \binom{N}{n}, & n=0,1,2,\cdots,c; \\ p_n/p_o = \left(\frac{N\lambda}{\mu}\right)^n \binom{N}{n} \frac{n!}{c!c^{n-c}}, & n=c,c+1,c+2,\cdots,N; \end{array}\right\} \quad (17)$$

where the value of p_o was to be determined from the requirement that

$$\sum_{n=0}^{N} p_n = 1. \quad (18)$$

*The present discussion of Palm's work [173] follows the summary given by Feller (see preceding footnote). Feller found it convenient to make use of general results obtained earlier in his book rather than completely retracing Palm's somewhat lengthy development from basic probability concepts (cf. [174]). Naor [161] states that "Feller (1950), following Palm's analysis, showed the problem of machine interference to be a special case of steady state 'birth and death processes.'" Thus, the time-dependent equations cited here should properly be attributed to Feller. However, the steady-state equations and solution are those which were originally obtained (in a different notation) by Palm. (See [174], pp.32-33.)

Palm also gave tables and graphs of productivity loss coefficients for machines and for repairmen. However, these were not based on explicit general formulae and had to be computed individually (recursively or as the sum of a finite series with terms involving the p_n) for each set of λ, μ, N, and c values considered.

The significant aspect of Palm's solution was the finding of a probability *distribution* for the state of the system as an intermediate step in the calculation of various quantities of interest. Earlier efforts (see preceding section) had been devoted toward more direct derivation of expected value formulae for quantities of interest. However, Palm's result in effect provided more (deduced) information about the stochastic nature of the system and it is not surprising that it served as a point of departure for many later investigations.

Other Early Investigations

Another theoretical solution was independently advanced by Ashcroft [3] in 1950. His model involved a single operator and a general distribution of servicing times, but otherwise his assumptions were the same as those listed above for Palm [173]. Ashcroft derived formulae for the average number of machines awaiting repair and for the operator utilization. He also suggested an economic criterion for selection of the optimum number of machines to assign to an operator. The application of these formulae to the special cases of constant repair times and negative-exponentially distributed repair times was shown and tables of the average number of running machines were presented for the constant repair time case. Similar tables for the negative exponential repair time case were published by Benson and Cox [13] in 1951.

Benson and Cox [13] gave solutions to a number of interesting problems involving negative-exponentially distributed machine interfailure times and repair times. The first essentially duplicated Palm's solution [173] for a single operative, but showed that the p_n and the machine availability could be expressed in very simple forms. The thesis of the paper was that the same general method could be used to solve more complicated problems. The application to the case of constant repair times was described, but it was noted that, due to its complexity, it had little advantage over the solutions of Khinchin [133], Westgarth [247], or Ashcroft [3] except that it led to the distribution of the number of stopped machines.

Formulae for the machine availability, operative efficiency, and the distribution of stopped machines were derived and comparative numerical examples were given for the following additional cases:

1. One operative and N machines where the machines are subject to two types of randomly occurring stoppages, each of which has a negative exponential distribution of clearing times (partial solution only).

2. c operatives and N machines where the machines are subject to c types of randomly occurring stoppages, each of which has a negative exponential distribution of clearing times and where each operative is a specialist who can treat only one type of stoppage.

3. Palm's problem [173] with c operatives.

The last part of Benson and Cox's paper [13] discussed adaptation of the results to cases involving operatives with ancillary duties. In an added note, Benson [11] more formally developed the concept. (This material will be reviewed in another section.) In still another note, Cox [33] gave a table for predicting the productivity of a large number of machines under the care of one operative. Table values were generated from a recurrence relation reported in the previous paper [13]. Benson [12] and Cox [38] have discussed part of their work in later papers. Portions of this work were also reviewed in Cox's expository paper [35] on the statistical analysis of problems of congestion.*

Takács [214], 1951, developed a model involving a single repairman, negative-exponential service time distribution, and arbitrary (general independent) failure time distribution.** Also in 1951, Muller [160] presented his doctoral thesis which dealt with the waiting time problem associated with attendance of a group of machines. In 1952, O'Connor [169] gave an example-laden discourse which reviewed much of the earlier factory-oriented investigations and described the use of machine interference theory in work measurement calculations. Some of the more interesting portions of O'Connor's paper were contributed by H. Ashcroft (see [3]).

In 1953, Benson, Miller, and Townsend [15] published a paper on

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*Cox has treated the statistical analysis of congestion much more extensively in two other papers [34,39]. The 1955 paper [34], which was partly motivated by Tippett's work [237], does in fact give a detailed presentation of statistical methods for the study of machine interference phenomena. The 1964 paper [39] is more generally oriented. Qualitative discussions of statistical approaches to machine interference problems have been provided by Tippett [238,239].

**More of Takács' work is described in a later section of this chapter. It would seem that the fresh ideas in Takács' paper [214] would have provoked the British-American mainstream of investigators to study repairman problems with more general input distributions. However, the paper, in Hungarian, apparently escaped notice. Takács paper [214] was not cited in a major bibliography until 1957 [45], and his methods did not attract widespread interest until after 1957 when Takács himself began publishing repairman papers in English.

applications of machine interference theory to production problems occurring in the textile industry. Much of their presentation was qualitative; however, they did extensively discuss and illustrate with numerical examples the methods of Benson and Cox [13] and Benson [11] for calculating production rates and work loads when the effect of interference is to be taken into account. An emphasis was placed on the treatment of ancillary duties, operator allowances, and machine interference allowances. Similar considerations were the object of much of Jones' work [109-115]. His expository article of 1956 discussed operator-assignment problems and work-measurement ramifications in considerable detail. A tutorial approach was adopted by Mitchell [155, 156] in presenting many of the same ideas.

Malcolm [150] and Mangelsdorf [151], 1955, studied various economic aspects of operator assignment schemes in machine interference problems. Malcolm discussed an application of Palm's multichannel model [173] to the economic organization design for a real-world production department engaged in engine testing. Mangelsdorf's paper, actually several chapters from his master's thesis, was an extensive discourse on the merits of existing machine interference models (encompassing most of the models thus far surveyed). Some points of special interest in his paper were discussion of limitations of the available model, cost analysis based on ratios between the cost parameter values, sensitivity analysis of machine assignment cost subject to variations from the optimum assignment, a demonstration of the economic benefits of multi-server assignments, and the many illustrations used to summarize results. A number of other authors stressed operator-assignment applications of machine interference theory during the period 1955-1957: Cox [37], Daru [41], Hénon [82], Van Dobben de Bruyn [242],* Kleinmann [135], Fetter [54], Knödel [136], and Buch [20]. Cox's paper [37] served to transmit to French audiences some thoughts from the British-American mainstream of theoretical developments.

More Complicated Models

Sewell [203] and Stoller [206], 1956, gave an unpublished discussion of a very general problem in machine maintenance. Stoller [207] later published some quantitative material from this discussion. As a special case of his general solution, Stoller gave the state probabilities for the repairman problem involving N machines and M operatives where (1) each machine is subject to c modes of randomly occurring failures; (2) the repairmen are divided into c teams of various sizes m_j $(j=1,2,\cdots,c)$ which specialize in servicing type j breakdowns--note that necessarily, $M \geq N$; and (3) each of the m_j members of a team services a type j breakdown according to the same (type j) negative exponential repair time distribution. It was noted that the subcase where $m_j = 1$ $(j=1,2,\cdots,c)$ had been solved previously by Benson and Cox [13].

Naor [161] has drawn together results obtained by Ashcroft [3], Benson and Cox [13], Feller ([52], pp.416-421), and Palm [173] and extended machine interference theory in several directions:

> It is the purpose [which was fulfilled] of this paper--following Palm's and Feller's analysis [and hence their

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*Van Dobben de Bruyn's paper [242] is cited secondhand from Saaty [196]; Doig [45] attributes the same paper to E. D. van Rest.

> assumptions]--to show (a) that the probability distribution function of the number of machines stopped may be derived in terms of Poisson functions; (b) that explicit, general expressions--in terms of Poisson functions--exist for all relevant quantities under consideration, such as average number of machines in working condition, loss due to machines in waiting lines, loss due to unoccupied repairmen, etc.; (c) that computational procedures are immensely simplified in numerical applications by the use of the tabulated Poisson functions; and (d) that the Ashcroft solution is completely identical with the Palm solution (for the one repairman case) if the appropriate prior assumptions are made, though, seemingly, two differing avenues of approach are being used by these authors in their mathematical analyses. ([161], p.281.)

The formulae derived in conjunction with purposes (a) and (b) were expressed in terms of an "S-function," closely related to the Poisson distribution function. A numerical example, using values from Molina's tables [157], showed how quantities of interest could be calculated in a much less tedious fashion than that employed by Palm.

In a further paper, Naor [162] noted two explicit shortcomings associated with his approach:

> (a) When the number of repairmen, r, is very large the evaluation of the S-function becomes incommodious . . .;
> (b) . . . On entering Molina's tables a parameter must be selected which, in our case, equals to the ratio of the number of repairmen r and the servicing factor. If the number of repairmen is large and/or the servicing factor is small, the ratio representing the Poisson parameter may exceed 100, the largest parameter of Molina's tables. In such circumstances the method proposed by the author in the above-quoted paper is of no avail. ([162], p.334.)

In order to remove these shortcomings, Naor derived approximate, limiting expressions in terms of the normal distribution function for cases involving large parameters "exceeding 100, say" and/or very many servers.

In a third paper, Naor [163] briefly summarized this work [161, 162] before proceeding with a discussion of the control of machine interference by an outside observer. This last paper has application to the case of patrolling repairmen and will be reviewed later, along with others on the same topic.

Peck and Hazelwood [176], 1958, have published extensive tables for use in repairman models involving N machines ($4 \leq N \leq 250$), M repairmen ($M<N$), and negative exponential inter-failure and repair time distributions. The table entries are a "service factor" X, an "efficiency factor" F, and a "delay probability" D. A brief theoretical preface defines these factors and demonstrates (with examples) their use in calculating average values of the usual quantities of interest.*

In his monograph on various aspects and applications of queueing theory, Morse ([159], Chapter 11) discussed some rather interesting consequences of simple machine interference theory. His observations were applied to the model originally formulated by Palm. For example, Morse contrasted the alternatives of using one repairman who performs repairs with an average time T_s or two repairmen who each perform repairs with an average time $T_s/2$, and demonstrated that use of the single, faster repairman would result in a smaller average number of nonoperative machines. Several useful graphs were given for the economic selection of repair-crew size. Although interesting, the chapter was expository in nature and did not serve to advance the theoretical state of the art.**

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*Since 1958, a number of finite-population queueing tables have been published. Descloux's tables [44] are to be particularly recommended for their ease of use.

**Initially, Morse considered the effect of a periodic preventive maintenance policy on a system involving one machine, one repairman, In 1959, Nasr [164] gave a discussion on some problems of machine interference. Some new computational methods were presented; however, Nasr's most significant contribution was probably the drawing together and showing of interrelations between the theoretical developments of Palm [173], Benson and Cox [13], and Naor [161,162]. Expositions considerably more extensive in terms of material covered and references cited were given, in 1961, by Saaty ([196], pp.323-332) and Cox and Smith ([40], pp.91-109,161-162) and, in 1962, by Takács ([228], pp. 189-204).

Devoted Repairmen, the Era Following Takács

Investigations in the Present Decade

Investigations of devoted repairman problems in the present decade have been primarily directed toward the extension of machine interference theory to models involving arbitrary (general independent) service-time distributions. Lajos Takács [219,220,222,227,228] has been the significant contributor in this area. Other recent work has been directed toward the development of new methodologies for the study of machine interference problems.

Earlier work, as discussed in the two previous sections, had developed formulae based on "expected-value" (not "state-probability") arguments which gave operator utilization and average machine

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and generally distributed failure and service times. Ostensibly, Morse's introduction of preventive maintenance considerations into repairman problems would appear to be a new development; however, regrettably, the analysis was not extended to multi-machine systems where *congestion* is a factor. Morse's problem was in fact a statement in particular physical units of a simple problem already "well-solved" in the renewal-theory literature.

availability for cases other than negative-exponentially distributed service times (e.g., see discussion of Ashcroft [3]). Further, some thought had been given to the use of negative-exponentially based formulae to approximate results for other cases. For example, Benson and Cox [13] had tabled

> . . . values of [average] machine availability for three distributions of clearing time, viz., constant clearing time, Pearson Type III distribution and exponential distribution of clearing times. (The values for the Type III distribution were obtained from Khintchine's solution [[133]]. It will be seen that . . . the differences in the values of [average] machine availability are not large. . . . ([13], p.75.)

Also, as Jackson and Adelson noted in their survey [100,101]:

> Even when lives are distributed as E_k [that is, according to the *kth* Erlang distribution], some calculations made by Adelson [[1]] indicate that an exponential life assumption does not lead to significant errors provided that the service-time distribution is represented fairly accurately by another E_k and that . . . [certain other conditions are met]. ([100], p.20.)

However, more sophisticated results did not appear until after the dissemination of Takács work on the state probability distribution for a system with generally distributed service times.

Takács' Work

Takács was active in the study of repairman problems as early as 1951. However, most of his extensive publications in the realm of stochastic processes* have been directed toward the study of (sometimes equivalent) telephone-traffic problems.** His first paper [214] on

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*Only a few of these will be mentioned in this survey. Reference is made to Saaty's bibliographies ([196], pp.404-405,413; [198], pp.470-471) and to the end-of-chapter references in Takács' books [228,231] for a comprehensive list (totaling several score!) of Takács' papers on stochastic processes.

**A number of these have application to repairman problems, with machine interference investigated the steady-state behavior of a single-server system in which the repair times were negative-exponentially distributed and the machine failure times followed an arbitrary (general independent) distribution.* Two contemporary papers [215,216] on telephony mentioned applications to machine interference phenomena.

In 1957, Takács [219] first published his derivation--it also appeared in his 1962 book [228] and has been widely quoted--of the state probabilities p_n $(0 \le n \le m)$ that there are n machines operative at any time in the m-machine, single-repairman problem. Each machine was assumed to have negative-exponentially distributed times from end of repair to next breakdown with mean time $1/\lambda$ and generally distributed service times with mean β and variance σ^2. Define

$G(t)$, cumulative service-time distribution;

r_{ij}, probability that j machines are operative just before the end of a service, given that i machines were operative just before the last service completion;

π_j, probability that j machines are operative just before the end of a service;

and let $\gamma(s)$ denote the Laplace-Stieltjes transform of $G(t)$, *viz.*,

$$\gamma(s) = \int_0^\infty e^{-st}\, dG(t), \tag{19}$$

Takács' argument proceeds in the steady-state, for which existence is assured since the system's state changes describe a finite, aperiodic

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and without spares, as shown by Barlow [5] whose adaptations are discussed in two places elsewhere in this survey.

*Takács' 1951 paper [214] had little impact on mainstream repairman-problem literature. See second footnote on p. 34.

Markov chain (imbedded at service completion epochs).

The probability that a particular machine, operating at time 0, fails during [0,t] is $1 - e^{-\lambda t}$. Using the binomial distribution, it follows that

$$\left.\begin{aligned} r_{ij} &= \int_0^\infty \binom{i+1}{j} e^{-j\lambda t}\,(1-e^{-\lambda t})^{i+1-j}\,dG(t), \\ &\qquad\qquad i=1,2,3,\cdots,m-2; \\ r_{m-1,j} &= r_{m-2,j}; \end{aligned}\right\} \tag{20}$$

for all integral j such that $0 \le j \le m-1$ and $j \le i+1$. Further,

$$\sum_{j=0}^{m-1} \pi_j = 1, \tag{21}$$

and

$$\pi_j = \sum_{i=j-1}^{m-1} r_{ij}\pi_i, \quad j=1,2,3,\cdots,m-1. \tag{22}$$

Let Q(z) be the generating function for $\{\pi_j\}$, *viz.*,*

$$Q(z) = \sum_{j=0}^{m-1} \pi_j z^j. \tag{23}$$

Then, from substitutions and simplification,

*The generating function for this discrete probability density function is just its geometric transform. Reference is made to Beightler, *et al.* [8] for one of the few good lists of these transforms.

$$Q(z) = \int_0^\infty (1-e^{-\lambda t}+ze^{-\lambda t})Q(1-e^{-\lambda t}+ze^{-\lambda t})dG(t)$$

$$+ (1-z)\pi_{m-1} \int_0^\infty e^{-\lambda t}(1-e^{-\lambda t}+ze^{-\lambda t})^{m-1}\, dG(t). \tag{24}$$

Now define the coefficients B_j by

$$Q(z) = \sum_{j=0}^{m-1} B_j(z-1)^j. \tag{25}$$

A term-by-term comparison of the two series (23,25) for Q(z) yields

$$\pi_n = \sum_{j=n}^{m-1} (-1)^{j-n}\binom{j}{n}B_j, \quad n=0,1,2,\cdots,m-1. \tag{26}$$

A similar comparison of the two series (23,25) for Q(z+1) yields

$$B_j = \sum_{i=j}^{m-1} \binom{i}{j}\pi_i, \quad j=0,1,2,\cdots,m-1 \tag{27}$$

which shows why B_j is referred to as the "j*th* binomial moment of $\{\pi_i\}$."*

From the second series (25), it can be seen that

$$\left.\begin{aligned} B_o &= Q(1) = 1; \\ B_j &= \frac{1}{j!}\left.\frac{d^jQ(z)}{dz^j}\right|_{z=1}, \quad j=1,2,3,\cdots,m-1. \end{aligned}\right\} \tag{28}$$

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*The approach introduced in this paper [219] has achieved some popularity in a variety of applications. It is called the "Method of Binomial Coefficients."

Applying this result to the integral equation (24) yields a second expression for the B_j in terms of the π_i. Then, solving for the B_j, one obtains

$$B_j = \frac{C_j \sum_{i=j}^{m-1} \binom{m-1}{i} \frac{1}{C_i}}{\sum_{i=0}^{m-1} \binom{m-1}{i} \frac{1}{C_i}}, \quad j=0,1,2,\cdots,m-1; \tag{29}$$

where

$$\left.\begin{aligned} C_o &= 1; \\ C_i &= \prod_{k=1}^{i} \frac{\gamma(k\lambda)}{1-\gamma(k\lambda)}, \quad i=1,2,3,\cdots,m-1. \end{aligned}\right\} \tag{30}$$

This completes the derivation for the π_n. Takács next derived expressions for the absolute probabilities p_n. The argument was similar, except transition probabilities p_{ij} were used instead of the r_{ij} and $\beta^{-1}[1-G(t)]dt$ was used in the integral instead of $dG(t)$ to indicate the remaining service time. Binomial moments for the p_n were introduced and related to those for the π_n. This comparison yielded the desired formulae for the p_n, *viz.*,

$$p_n = \frac{m\pi_{n-1}}{n(m\beta\lambda+\pi_{m-1})}, \quad n=1,2,3,\cdots,m; \tag{31}$$

and

$$p_o = 1 - \sum_{n=1}^{m} p_n. \tag{32}$$

Takács then showed that the average number in the system is

$$\frac{1 - p_m}{\lambda\beta} = \frac{m}{m\lambda\beta + \pi_{m-1}} ; \tag{33}$$

showed that the average waiting time of a machine is

$$\beta(m-1) + (1-\pi_{m-1})\left[\frac{\sigma^2 + \beta^2}{2\beta} - \frac{\beta}{1 - \gamma(\lambda)}\right] ; \tag{34}$$

and obtained many other results of interest for the stationary process. Takács studied the same problem in three other papers [220,222,227], although telephone traffic was the primary subject in each. Review material is available in his books [225,228].

Other Investigations

Harrison [80], 1959, has taken an independent approach to the same problem [219], Harrison noted that the length of (number of units in) the waiting line at instants of service completion formed a Markov chain. Using Kendall's method [129] of the imbedded Markov chain,* he was able to obtain probability density distributions of the waiting-line length and the waiting time. As a demonstration, the results were specialized to the cases of negative-exponentially distributed and constant service times.

Barlow [5],** 1962, in an expository paper, sought to exploit the

*See also Gaver [66].

**The original version of Barlow's paper is typographically confusing in references to authors and equations. A more readable, but relationship between repairman problems and the more general realm of queueing problems. Three problems treated were:

1. Negative-exponentially distributed failure and service times for each machine, m machines, and m repairmen;

2. Arbitrarily distributed failure and service times, one machine, and one repairman;

3. Negative-exponentially distributed failure times, arbitrarily distributed service times, m machines, and one repairman (i.e., Takács' problem [219] discussed above).

The first problem was solved using general birth-and-death theory arguments advanced by Karlin and McGregor [118-121,123]* and Harris [79]. It was observed that the system is that of the continuous Ehrenfest model [47] for diffusion processes.** Time-dependent transition probabilities $P_{in}(t)$ were stated (from [121]) and the limiting (as $t\to\infty$) probabilities p_n were deduced. The second problem was identified as an elementary semi-Markov process with renewal theory implications. Various expected values and distributions were given, including some results due to Takács [215,220]. The third problem was observed to have an imbedded semi-Markov process. The stationary solutions were quoted from Takács [219]. Barlow also discussed repairman problems with

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less thoroughly documented version appears in the text by Barlow and Proschan ([6], pp. 139-151).

*Karlin [116] and Karlin and McGregor [117,122,124] have published other works that are relevant in the present context. An early interest in generalized birth-and-death theory was taken by Kendall [126], 1948.

**See, e.g., Feller ([52], pp. 343-344,358; [53], pp. 450-454) for discussion and additional references.

spare machines. This portion of his paper will be reviewed in Chapter III.

Blom [17], 1963, studied Takacs' problem [219] from the point of view of the operator. The operator's total time can be divided into busy periods and free periods. Referring to the sum of a free period and a busy period as a "work cycle," Blom obtained the joint Laplace transform of three random variables: (1) duration of the busy period, (2) number of repairs performed during a busy period, and (3) total productive time of the machines during a work cycle. Blom also briefly considered a more general time period than the busy period, "namely a period beginning at a moment when a of the n machines are standing still and the operator has just begun to serve one of these and ending when the operator becomes free again."

A paper by Thiruvengadam and Jaiswal [234] had a similar orientation, but the object of investigation was the simple machine interference problem (i.e., Palm's problem [173]). They derived an expression for the Laplace transform of the distribution of the duration of a busy period. It was shown that, under suitable limiting conditions, the average duration of the busy period for the simple machine problem corresponds to the average duration of the busy period for the infinite-population case. In another paper [235], 1964, they applied a modified version of Takács' formulation [219] to solve Takács' problem [219]. The modification enabled them to obtain the distribution of busy periods as a particular case. Portions of the formulation and solution were based on the work of Keilson and Kooharian [125]. Incidentally, Jaiswal has also presented several papers [103-106] which, in context, reviewed aspects of machine interference theory.

Hodgson [86], 1964, also has studied Takács' problem [219]. His interest centered on the distribution of the duration of down time of a machine. The author suggested that his "methods and results are a little simpler than those of Takács." An expression for a distribution of non-standard "waiting times" was obtained in the case of negative-exponentially distributed repair times.

In a recent (unpublished) paper, Whitehouse [248] has discussed the solution of finite queueing problems by graphical means. His approach involved the representation of a model as a GERT (Graphical Evaluation and Review Technique) network. By solving the network in various ways, Whitehouse was able to find the moment generating function of a number of probability distributions of interest. A repairman model was used as an example.

Analogue Simulation of Repairman Problems*

The first analogue computer for simulation of machine interference was apparently the simple device invented by Jones [110,111], 1949.

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*This chapter excludes a *grouped* survey of digital computer simulations of repairman problems; the significant papers in this area are discussed elsewhere in this survey according to the type of model studied. Digital simulation studies are normally a "second choice" approach to problems for which the theoretical models are intractable or nonexistent. The formulation techniques are well known and a discussion of them here would not contribute to the present purpose of outlining the development of machine interference *theory*. The analogue approaches discussed in the present section are included because of their intrinsic interest or because they have been closely associated with the mainstream of theoretical developments. See Page [172] for a review and critique of the use of computers in the study of queueing problems. Green's paper [72] may also be of some interest since it

It was applicable to situations involving a repairman servicing, as necessary, up to ten machines in rotation.* The computer contained a timing and revolution counting device which provided the basic data necessary for computation of the servicing, running, and interference times of the machines for each test.

Palmatier [175], 1956, constructed a small electronic analogue computer to use as a pilot model in testing the feasibility of using such devices to solve complex machine interference problems. The results were compared with certain theoretical predictions.

Dunn, Flagle, and Hicks [46], 1956, constructed an electro-mechanical analogue for the simulation of queueing network problems. Called the QUEUIAC, it features paper teletype-tape input for service and exogenous arrival distributions, continuous display of queue-length information, and capability of simulating a number of flow-merger and queue disciplines (including priority servicing). It was stated that

> In all cases the rates of tape input may be controlled manually and, in some cases, they may be controlled automatically to simulate feed-back features of certain handling operations. ([46], p. 650.)

No examples were given; however, it was apparent from the capabilities description that a combination of (proportional) manual arrival-rate control and automatic service-rate control could be used to simulate

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contains a GPSS (General Purpose Simulation System) digital computer simulation program used in the study of a variation of a repairman problem.

*A number of investigators have advanced analytic models of this servicing scheme. See the discussion of "patrolling repairmen" which appears later in the chapter.

repairman problems involving very general failure and service distributions, m machines, and c repairmen. Schematics of the QUEUIAC's components were included in the paper.

Huggins [93], 1960, has given examples showing how a "system dynamics" approach can be used to obtain time-dependent state probabilities for repairman problems. The cases treated were Palm's model [173], as reported by Feller ([52], pp. 416-421), of a single repairman and (variously) one, two, or five machines. Huggins' approach consisted of (1) recognizing that the system dynamics were Markovian; (2) expressing the Markovian interrelations directly in (Laplace-transform domain) flow-graph form--there was no need to develop formal equations;* and (3) solving the flow graph by (a) manually calculating the appropriate graph transmissions and inverting the resultants into the time domain, or (b) transferring the flow graph to a (standard, commercial) analogue computer to measure the state probabilities as functions of time.

An intriguing aspect of Huggins' paper was the indication of equivalence between the flow-graph and analogue-computer representations. This equivalence proved to be quite profitable: In the five-machine problem, he used techniques of flow-graph theory** to first

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*In step (1), Huggins did not include an explicit statement of (continuous-time) *Markovian behavior;* however, this is implicit from the nature of the background material cited and his easy accomplishment of step (2).

**A knowledge of flow-graph theory is evidently of some merit in the investigation of repairman problems. This knowledge is not provided in Huggins' paper [93]. The interested reader should refer to his previous paper [92] *and* to the references cited in both papers, especially the work of Mason, Lorens, and Howard. Additional references are cited by Gue [74]. If possible, the reader should also review Howard's unpublished manuscripts [89,90] which contain a lucid and extensive reduce the graph to a very simple form. The corresponding analogue-computer representation then required only six integrators and one inverting amplifier to solve the problem.

Gue [74], 1966, has extended and formalized the notion of correspondence between flow graphs and analogue computers. His paper contains a table of analogous flow-graph and analogue-computer elements and several illuminating, comparative examples. His presentation is not restricted to repairmen problems, but applies to finite queues and finite Markov processes in general.*

Repairmen with Ancillary Duties

Machine interference problems involving operators with ancillary duties--that is, operators with duties other than the starting of stopped machines--have been of interest primarily to the textile and other automated production industries. Cox and Smith [40], in an account based on other work [11,13], have indicated the motivation behind and the approach taken to many industry-oriented studies:

> In many practical applications the most unrealistic assumption in the analysis of section 4.2 [which was a discussion of devoted operatives] is the one that the operative is always available to restart stopped machines, that is that the restriction on service is purely one of capacity and not one of availability. In fact, in machine interference problems,

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account (and examples) of flow-graph analysis--both steady-state and time-dependent--of Markov and semi-Markov processes. Methods of altering flow graphs to permit direct calculation of some 30 steady-state statistics of interest for a system are also given by Howard.

*Syski [213] has given a unifying treatment of finite Markovian queues. His paper contains some 70 references to other work in the field. See especially § 3.3 for a discussion of queues with input from finite populations.

> some allowance must always be made for relaxation and personal needs, and quite often also the operative will have other duties, such as fetching raw material. . . . A full discussion of the effect of ancillary work on the rate of production would require a knowledge not only of the rate of incidence and duration of periods of ancillary work, but also of the priorities to be given to work of different types.* It is nearly always impossible to specify these things at all precisely. *Therefore, the most sensible thing is to look for semi-empirical rules for modifying the solutions of section 4.2.*** ([40], pp. 102-103.)

Apparently, most investigators have done the "sensible thing" since there exists in the literature a preponderance of such semi-empirical rules and a corresponding paucity of formal theoretical models for treating cases of repairmen with ancillary duties.

The majority of these semi-empirical rules are oriented toward practical operator-assessment and work-measurement activities in production systems*** and seemingly can contribute little to the mainstream of theoretical developments in machine interference theory. The work of Benson and Cox [13] is a notable exception, perhaps, because of its very simplicity and lack of operator involvement.

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*Papers treating multi-purpose servers and arrival priorities will be reviewed in a later section of this literature survey.

**Italics added.

***A detailed discussion of these semi-empirical rules for adjusting devoted-repairman statistics to reflect the effect of ancillary work is beyond the scope of this survey. The interested reader may refer to Jones [115] for a thorough discussion of work measurement, operator compensation, personal and fatigue allowances, etc., in multi-machine assignments. See also the papers listed under Code 130, p. J-4, in "Index to *The Journal of Industrial Engineering,* Volume I 1949 through Volume XVIII 1967" (American Institute of Industrial Engineers, New York, N.Y.). A less involved discussion is given by O'Connor [169]. The paper by Benson, Miller, and Townsend [15] may also be of interest. It gives a non-mathematical account of the problem and discusses in detail, with examples, a number of applications in the wool-textile industry. For other industrial applications see Mangelsdorf [151] and additional references cited by Hillier [84].

Benson and Cox [13], 1951, have obtained from general, but reasonable (as proved by Benson [11]) arguments some very simple rules for the treatment of special cases of ancillary duties. For all cases of "loose ancillary work," which the operative can arrange to do during periods when all machines are running, they gave the rule:

> If a fraction L of the operative's time has to be spent on loose ancillary work, then--(i) the maximum rate of production is $(1-L)/x$; (ii) to find the minimum number N_0 of machines to attain this rate of production, find the value of N in the tables that gives operative efficiency $(1-L)$. ([13], p.80.)

Here N is the number of machines and $1/x$ is the maximum rate of production for the devoted repairman case.

For "fixed ancillary work," which the operative must do at certain times regardless of the state of the machines, Benson and Cox gave rules for only two cases, "concentrated" and "spread" fixed ancillary work:

> If a fraction C of the operative's time must be spent on concentrated ancillary work, the machine availability and rate of production are $(1-C)$ times those deduced from the tables. ([13], p.80.)

> If a fraction S of the operative's time is spent on spread ancillary work, define a new value $x' = x/1-S$. Provided that the operative efficiency is near 1, it and the machine availability are found by entering the Tables at x' instead of at x. ([13], p. 80.)

Concentrated fixed ancillary work consists of infrequently occurring ancillary work which is of long duration compared to the inter-stoppage time of a machine. The rule reflects a neglect of production occurring while the ancillary work is being done. Spread fixed ancillary work consists of a large number of very short ancillary work periods spread approximately uniformly in time. The rule reflects a regard of the mean clearing time as being increased by the factor $1/(1-S)$.

Benson and Cox [13] also gave two examples illustrating the use of the rules. One example involves an operative who has several different types of ancillary duties.

In a further note, Benson [11], 1952, gave an analytical formulation of a repairman problem involving ancillary work. A single operative's time was regarded as being made up of periods of work on the machines ("machine periods") and away from the machines ("external periods"). Each of the N machines were assumed to have negative-exponentially distributed stoppage and clearing times with parameters γ and c, respectively. To install a measure of generality into the solution, Benson assumed that the durations of machine and external periods have chi-square distributions with 2ρ and 2σ degrees of freedom,* respectively, and with mean values ρ/m and σ/a, respectively.

Using a device suggested by Kendall [127],** he developed a number of stage-descriptive ($r=1,2,\cdots,\rho$; $s=1,2,\cdots,\sigma$), steady-state, recurrence relations among various state probabilities and involving the parameters γ, c, r, ρ, s, σ, m, and a. The stage parameters, r and s, were eliminated by appropriate summations to obtain finally

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*Note that the *kth* Erlangian distribution is a chi-square distribution with 2k degrees of freedom.

**Kendall's device is an application of the "method of stages," originally devised by Erland [18] to represent non-negative-exponential distributions in terms of negative exponential distributions. Let x_i $(i=1,2,\cdots,k)$ be a negative-exponential variate with mean $1/\mu$. Then $y = x_1 + x_2 + \cdots + x_k$ is distributed according to the *kth* Erlangian distribution and has mean k/μ.

$$cP_M(\nu) = (\nu+1)\gamma\{P_M(\nu+1) + P_E(\nu+1)\}, \tag{35}$$

where $P_M(\nu)$ is the probability that ν machines are running during a machine period and $P_E(\nu)$ is the probability that ν machines are running during an external period. This last equation was supported by several other relationships which were restricted to special cases of the parameters.

Benson was not able to obtain a general solution for the $P_M(\nu)$ and $P_E(\nu)$.

> A general solution of these equations is difficult to obtain but the equations have been solved for two cases, (i) when the durations of both machine and external periods have negative exponential distributions (i.e., $\rho=\sigma=1$), and (ii) when the durations of machine periods have a negative exponential distribution but the duration of external periods is constant (i.e., $\rho=1$, $\sigma\to\infty$). ([11], p.202.)

In addition, Benson validated the extreme-case rules obtained by general arguments in the previous paper [13] by showing them to be limiting cases of the formally derived solution. A confined report of results obtained by Benson [11] and Benson and Cox [13] appeared, with examples taken from Benson, Miller, and Townsend [15], in the 1961 monograph by Cox and Smith ([40], pp. 102-106).

Rosenberg [193], 1963, treated a case of ancillary work involving one repairman, N machines, negative-exponentially distributed failure times, and generally distributed service times. In Rosenberg's model, the repairman services machines in the usual way until such time as there are no machines waiting for repair. He then departs for a fixed time C to perform ancillary duties. Upon returning, he begins servicing failed machines as before (remaining idle until a machine fails if no failures have occurred during his absence). Rosenburg's analysis led to stationary probabilities for the number of working machines.

More recently, Thiruvengadam [233], 1965, has approached the problem of ancillary work with the use of discrete transforms to obtain closed-form solutions for (1) the distribution of the number of machines awaiting repair and (2) the stochastic law of the busy period. A single server, negative-exponentially distributed interfailure times, and arbitrarily distributed service times were assumed. In Thiruvengadam's model, occurrence (stochastically) of ancillary duty during repair of a machine would result in the operator's immediate departure, with service being resumed on the same machine upon his return. Thiruvengadam's results are very similar to those obtained in two related papers [107,236] by Thiruvengadam and Jaiswal on machine interference with priority servicing. These will be discussed in a later section.

Machine Interference with Overnight Repair

Elce and Liebeck [48], 1966, have investigated a machine interference problem in which an intermittently operating group of machines is made fully operative before each interval of system activity. Their model involved negative-exponentially distributed repair and failure times, N machines, and a single repairman. Each work period of fixed length T starts at time t=0 with all machines in working order. During the interval [0,T], the repairman services failing machines in the usual way. At time t=T, all work ceases and any machines which were then nonoperative are restored to full working order before a new work interval is started. Elce and Liebeck obtained expressions for (1) $a(t)$, the expected number of machines working at time t; (2) upper and lower bounds for $a(t)$; (3) upper and lower bounds for the expected number of machine hours worked during each interval $[0,T]$.

Cooperating Repairmen

Bawa and Nair [7], 1966, have constructed a model which permits the calculation of a machine's expected waiting time for service when a group of N machines is tended by c repairmen who cooperate according to a particular, well-defined, state-dependent service scheme. Several repairmen may cooperate on a service call only in those states where no machine is waiting for service, and a new dispersement of repairmen results whenever the system changes state. The failure times of a machine are assumed to be negative-exponentially distributed as are the service times; however, for the latter the mean service rate depends on the state of the system and the cooperative service policy.

Using an example from Palm's paper [174], Bawa and Nair demonstrated how one could determine the optimum number of cooperating repairmen to assign to N machines according to certain economic and effectiveness criteria. This solution was contrasted with Palm's solution for the analogous problem involving non-cooperating repairmen. The comparison presented a strong case for the use of cooperating-repairmen assignments.

Patrolling Repairmen

A number of investigators--Runnenburg [194]; Mack, Murphy, and Webb [149]; Mack [148]; Howie and Shenton [91]; Ben-Israel and Naor [9,10]; and others--have studied situations where a repairman observes a regular system of patrolling.* In such cases, the repairman may be thought of as visiting each machine in a prescribed cyclic order, spending a time t_1 in walking between machines, a time t_2 with machines that are running, and a time t_3 with machines that are nonoperative. Regular patrolling schemes are often advantageous when machines are distributed over a large area and/or it is important to examine each machine frequently. Their disadvantage lies in the fact that a stopped machine may remain down for a long time even when no other machines are stopped.

Formal work on machine interference problems involving systematic patrolling was preceded by the empirical efforts of, for example, Stribling [209] in 1952 and Hahn [76] in 1955. An early discussion of a patrolling operator was given by Brunnschweiler [19] in 1954; however, the first model of general significance was probably that presented by Mack, Murphy, and Webb [149] in 1957.

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*The patrolling-operator model was originally conceived and developed for a textile-industry application. However, it also has parallels in maintenance problems associated with fleet-type transportation systems. For example, as described in a paper by Ewashko and Baglow [51], the Mid-Canada Line (railroad) uses a helicopter to transport a repair crew between several isolated sites to perform preventive and other maintenance in accordance with a specified program. Although Mid-Canada's problem is somewhat more complicated than those treated here, the same basic considerations of service times, travel times, and visit sequencing are involved.

The assumptions of Mack *et al.* were (1) a negative exponential stoppage time distribution with parameter γ for each of N machines; a constant time k_i for the operative to walk from the (i-1)*th* to the *ith* machine and inspect and service it, before repairing it or proceeding to the next machine; and (3) an additional constant time R to repair and restart any stopped machine. Let q_n denote the probability that precisely n machines required repair on a tour of all machines and define K by

$$\sum_{i=1}^{N} k_i = NK. \tag{36}$$

Noting that the probability that the machine with which the operative started a cycle is still running at the end of that cycle is $e^{-\gamma(NK+nR)}$, Mack *et al.* established the recurrence relation

$$q_n \, e^{-\gamma(NK+nR)} + q_{n+1} \, e^{-\gamma[NK+(n+1)R]} = q_n, \tag{37}$$

which was shown to have the unique solution

$$q_n = (a^{n-1}b-1)(a^{n-2}b-1) \cdots (ab-1)(b-1)q_o \tag{38}$$

where $a = e^{\gamma R}$, $b = e^{\gamma NK}$, and q_o is determined from

$$\sum_{n=0}^{N} \binom{N}{n} q_n = 1. \tag{39}$$

This result enabled Mack *et al.* to determine formulae for the average duration of a round, the average number of machines found stopped during a round, the fraction of time an operative spends on repair, and machine running efficiency. A table of machine running efficiency (i.e., "the percentage of running time to total (stopped + running) time") for various values of N, γR, and γNK was also given.

Mack [148] extended these results to the case where the machines may break down stochastically in any of M ways, each way j ($j=1,2,\cdots,M$) having a negative exponential inter-occurrence time density function with parameter γ_j and an associated constant repair time R_j. The model was formulated and solved in a manner similar to that employed in the previous paper [149]; however, "the probability that the *ith* machine from the reference and no other is found stopped on a round is not independent of i in general as it is when repair times are constant . . ., and this causes some difference in the treatment of the two cases" ([148], p. 173). Mack described a conversion of his model to one involving repair times which are continuously distributed, so that his paper in fact extends the previous model [149] to the case where repair times have a density distribution of a very general nature.

It is worth noting that the essential difference between systematic patrolling models and the conceptually (but not mathematically) simpler models previously discussed is inherent in the walking time effect and not in the queue discipline. Witness:

> Again, our formulae for the case of zero walking time agree with those of Ashcroft, despite the difference in assumption about the order of dealing with the machines when more than one is stopped. This is to be expected when repair times are constant,

> for it is immaterial which of several stopped machines is started first. ([149], p.167.)

> As far as I am aware only Ashcroft (1950) has treated generally distributed repair times and he dealt only with the efficiency in the zero walking case. His solution and that given in this paper are the same when N = 2,3, and 4, and it seems probable that this is true generally though it is not easy to demonstrate. ([148], p.173.)

Howie and Shenton [91] have approached the systematic patrolling problem from a different viewpoint. They studied a single machine under the assumption that it (1) stops when total running time reaches a fixed value T; (2) has accidental stops according to a uniform probability distribution; and (3) is patrolled, that is, inspected and restarted if necessary, at fixed intervals P which may be less than T. They were able to express in terms of Laguerre polynomials the probability distribution of the number of patrols required to exhaust the machine's total running time. A number of properties of the distribution were illustrated by an example and its asymptotic normality as T increases was established.

Ben-Israel and Naor [9] have presented a discourse on systems in which a repairman attends a number of machines, devoting some of his time to ancillary duties, inspection, and walking. Throughout the investigation, it was assumed that direct attendance time on a machine was independent of the state of that machine. This assumption served to focus analytical emphasis on the distribution of inter-machine "walking" times. The various models discussed were differentiated according to the following considerations: (1) successive (ordered) versus simultaneous attendance to a group of machines; (2) constant Poisson breakdown intensity of the whole system versus constant Poisson breakdown intensity of individual, operative machines; (3) different assumptions regarding attendance times, including the cases of operator interarrival (round) times following constant, exponential, gamma, or general probability distributions. In all, 16 major models and their ramifications were discussed. Formulae were obtained for the first two moments and, in most cases, for the probability distribution of (1) the number of stopped machines observed by a random controller, (2) the number of stopped machines observed by the repairman during one round, and (3) the waiting time until repair of a stopped machine. Much of this information was summarized in a convenient tabular form.

Naor [163] had previously studied simultaneous service under the heading of "Ends Down" models. He derived various mean, variance, and density function formulae for the number of stopped machines as observed by a random controller and as observed by the repairman. As only his final results are quoted in the latter paper [9], both papers should be read together.

In an extension to their previous paper [9], Ben-Israel and Naor [10] studied a system which had elements of both simultaneous and successional attendance to machines. Their model assumed that machines are serviced in groups and that repair of machines belonging to the same group is simultaneous while different groups are dealt with successively. Formulae were derived for the mean and variance of the number of stopped machines as observed by a random controller and as observed by the repairman.

Tsubaki [241] has applied calculus and graphs, rather than stochastic birth-and-death theory, to discuss the optimum assignment of operators who attend spinning frames while walking along a fixed route. He studied the effect of varying the number of operators and/or the machine speed on the total cost of labor, production loss, and material waste. An approximate solution was obtained and the results were presented in various graphs, from which one could deduce the optimal strategy.

Conway, Maxwell, and Sampson [32] have also used a non-queueing theory approach to the study of a patrolling repairman. First, they developed equations showing various time requirements made up of deterministic and stochastic elements. Then, using a Monte Carlo simulation to generate values for the random variables, they approximated the solution to these equations.* The procedure was demonstrated on an example involving three machines and a single operator. Also presented was an ALGOL program for computer solution which showed man- and machine-idle time, production, and mean cycle length as outputs.

Repairman Models with Priority Servicing

Benson and Cox [13] apparently were the first investigators to study a machine interference problem where the order of service of failed machines follows a given priority discipline. The assumptions of their model were:

1. One repairman and N machines.

2. Each machine is subject to two different types of randomly occurring failures.

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*See Page [172] for a review and critique of the use of computers in the study of queueing problems.

3. Each type of failure requires a service time that follows a negative exponential distribution. That is, there are two types of service which correspond respectively to the two types of machine breakdowns.

4. Failures of the first type are always repaired first, but this priority is "nonpreemptive" in the sense that arrival of a high priority job cannot displace a lower priority job which is already under way.

Benson and Cox obtained equations which interrelated the stationary probabilities of possible states of the system. While they were unable to obtain a general solution to the set of equations; they did report the results of numerical solutions for several examples involving small values of N. In comparing the relative advantage of assigning a higher priority to type 1 (type 2) failures which characteristically require shorter (longer) service times, the authors concluded from their limited results that

> It appears to make very little difference whether long or short stoppages are cleared first. This may be considered to be a surprising result, since one would imagine intuitively that production would be increased by attending first to machines which can be restarted quickly. We suggest that the reason for the small difference is that by neglecting long stoppages the chance of interference is increased. ([13], p.77.)

It may be of interest to note that the intuitive notion does appear to hold in the case of an infinite population of machines ([27]; [178], p. 76).

A major breakthrough was achieved in 1954 with the publication of Cobham's pioneering work [28,29] on priority queueing for a (parametrically) general priority system. Cobham formulated a model involving r classes of nonpreemptive priorities, Poisson input with parameter λ_j (j=1,2,$\cdots$,r) for members of the j*th* class, negative-exponentially distributed service times with parameter μ_j (j=1,2,$\cdots$, r) for members of the j*th* class, and multiple channels. The model was generalized to allow arbitrary service time distributions in the case of a single channel. He obtained explicit expressions for the expected waiting time of a unit belonging to the j*th* priority class. Cobham mentioned that his results had application to machine maintenance, but did not himself explore the finite population problem.

Cobham's work served to set the mood and object of later investigations of priority queueing by demonstrating that fruitful analysis could be applied to obtain expected-value characteristics of various waiting time phenomena.* A strong argument for its significant influence on the study of priority queueing disciplines in repairman problems is indicated by two trends in the literature, (1) with Cobham's work as a font, the mainstream of analytical developments in priority queueing has dealt with waiting time phenomena for infinite population cases; and (2) repairman models with priority queueing have usually arisen as direct analogues to models developed for infinite population cases.

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*Except in very simple cases of priority queueing, the difficulty associated with the usual object of seeking state probability distributions is that, with a priority queue discipline, either (1) a cumbersome number of states and parameters of the system arise in a straightforward formulation of a model such that a subsequently attempted solution becomes frustrating (e.g., as in Benson and Cox [13]), or (2) attempted formulation of a less detailed model with fewer and more directly pertinent states becomes frustrating since transitions take on non-Markovian characteristics.

Holley [88] simplified Cobham's solution. Phipps [178] unified Cobham's results for the saturated [28] and unsaturated [29] cases of the single channel model, reported several results from Cobham's unpublished paper [27], and generalized Cobham's results for the single channel model to a continuous number of priorities with application to machine repairs. However, Phipps' paper had only limited applicability to repairman problems.

> From a practical viewpoint the results obtained are limited in applicability by the assumption of an infinite population of operating machines (i.e., of potential repair jobs). Our results will serve, however, as a guide to further thinking about priority assignment in machine maintenance, and as a basis for approximate quantitative study of problems involving very large numbers of machines [or very small servicing factors λ_j/μ_j]. ([178], p.77.)

In a short comment on Phipps' paper, Van Voorhis [243] urged the explicit consideration of costs in the evaluation of alternative priority assignments.

Quite a number of other papers on priority queueing with infinite populations have influenced or served as partial solutions to repairman problems with priority queue disciplines. In particular the work of Heathcote [81], Keston and Runnenburg [131,132], Miller [152], and Jaiswal [102] has also been cited in this connection.*

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*In order to properly trace and assign theoretical and methodological credit for priority queueing developments in machine interference theory, it would be necessary to comprehensively survey the mainstream of such developments for the infinite population case. This is beyond the scope of and would detract from the emphasis of the present discussion. The reader is referred instead to Avi-Itzhak and Naor [4] and Gaver [67] for reviews and bibliographies, to Takács [230] for a 'drawing-together' of early results, and to Avi-Itzhak and Naor [4] for a presentation of general features of 14 models and of their analyses. Koenigsberg's paper [137] on nonpriority special service is also of interest.

Jaiswal and Thiruvengadam have been relatively active in the study of repairman problems with priority queueing disciplines. In one paper, Jaiswal and Thiruvengadam [107] considered simple machine interference with two types of failure. Their assumptions were the same as the four listed above for Benson and Cox [13], except that service times were allowed to follow an arbitrary distribution rather than being restricted to the negative exponential case. Using the supplementary variable technique, the authors obtained the steady-state probabilities for the number of machines awaiting repair. Their model was described as the finite-source analogue of the problem treated by Miller [152] and Jaiswal [102] (portions of the analysis were based on Jaiswal's work [102]); was shown to reduce to Takács' solution [228] for the case of one type of failure; and was shown to yield a general solution to the problem formulated by Benson and Cox [13]. Also, the authors reported some numerical results for cases of negative-exponentially distributed service times and concluded, as had Benson and Cox, that

> From the limited numerical results presented . . . it appears that there is no significant difference in O. E. [operative efficiency] and M.A. [machine availability] whether long or short failures are repaired first. ([107], p.635.)

In another paper, Thiruvengadam and Jaiswal [235], 1964, gave a general discussion of the use of the supplementary variable technique (see Cox [36]) and discrete transforms in the study of machine interference problems. It was noted that this method had been adapted (e.g., as in [107]) to study more complicated problems involving two types of machine failure; and that it could be used to study two-source, priority-queueing processes in which either or both sources consisted of a finite population.

In a third paper, Thiruvengadam and Jaiswal [236], 1964, studied a variation of their previous problem [107] in which the priority discipline was "preemptive-resume" rather than being "nonpreemptive."*
For this problem, Thiruvengadam and Jaiswal formulated a model which allowed them to obtain expressions for the operative efficiency and machine availability.

More recently, Hodgson and Hebble [87], 1967, have considered a model which is the finite-population analogue of the classical priority-queueing model of Kesten and Runnenburg [131,132]. Hodgson and Hebble's model is based on the following assumptions:

1. One repairman tends k batteries of machines with m_i machines in the *ith* battery ($i=1,2,\cdots,k$). He can repair but one machine at a time.

2. The queueing discipline is nonpreemptive priority, with machines in the *ith* battery having priority over those of the $(i+1)$*th*.

3. The failure time for a machine in the *ith* battery is negative-exponentially distributed with parameter λ_i ($i=1,2,\cdots,k$).

4. The service time for a machine in the *ith* battery is

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*The "nonpreemptive" discipline was described earlier in the survey. Under a "preemptive-resume" discipline, a newly arriving high-priority unit may displace a lower-priority unit from the service facility, take its place, and receive immediate service. Service is resumed on the low-priority unit without loss of accumulated service time when the high-priority unit departs. The "preemptive-repeat" discipline is similar to the "preemptive-resume" except that a displaced unit loses any accumulated service time and must begin service anew.

generally distributed with distribution function, say, $S_i(t)$ ($i=1,2,\cdots,k$).

The authors formulated and solved a set of equations for the multivariate binomial moments of the number of operative machines in each battery immediately following a service completion. The analysis was based on Takács' method [219] of binomial moments.

Bibliographical Sources

Bibliographies on repairman problems, *per se*, do not exist in the literature. This fact undoubtedly explains why the dissemination of existing knowledge has been so irregular. The interested researcher must instead take recourse to the general bibliographies on queueing theory. Those which proved particularly helpful in conducting this literature search include the bibliographies compiled by Doig [45], Kendall ([128], pp.171-173; [130], pp.13-15), Riley [187,188], Saaty ([195], pp.197-200; [196], pp.375-413; [198]), Smith and Wilkinson ([204], *passim)*, and Takács ([228], *passim)*. The bibliography of Lunger [147] proved inaccessible; however, his list is apparently incorporated into Saaty's major bibliography (see [196], p.374).

CHAPTER III

A SURVEY OF THE LITERATURE ON REPAIRMAN MODELS WITH SPARES

Introduction

This chapter will consist of a review of the significant literature on repairman models with spares. Topics to be treated include repairman models with spares; reliability-oriented variations of the repairman model with spares; cyclic queues; and networks of queues.

The literature on repairman models with spares is severely limited. Only a few papers were found and one of these duplicated earlier work. It is thought that the value of these papers lies not in the models presented nor in the techniques demonstrated--they are too limited in scope to serve as an underlying methodological basis for extensive new work in the area--but in the indication of suitable means for adapting some of the regular repairman models presented in Chapter II to more complicated problems involving spares provisioning.

The repairman models with spares of Taylor and Jackson [232] and Toft and Boothroyd [240] are cases in point. It will be seen that they are expressed in a form similar to that of the regular model of Palm [173]. It is suggested that the many (regular) variations on Palm's model which were discussed in the previous chapter can easily be adapted to yield analogous variations for the models of Taylor and Jackson and Toft and Boothroyd. (See Chapter IV.)

A common assumption in the vast majority of the papers reviewed is that steady-state conditions prevail. The reader may find it helpful to adopt this assumption as a convention; all exceptions will be clearly identified.

Characteristics of Repairman Models with Spares

The flow system of a repairman model with spares is illustrated in Figure 4.

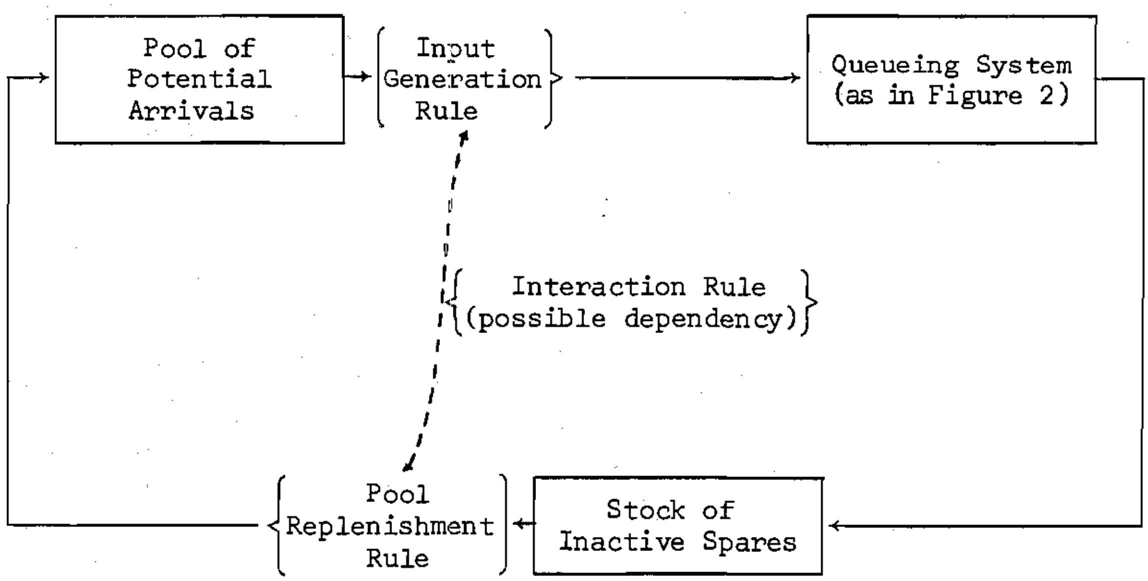

Figure 4. The Repairman Model with Spares

The "input generation rule" operates in the same way as that for the regular repairman model of Figure 3. The "pool replenishment rule," at least for the models discussed in this chapter, operates to keep the pool of potential arrivals (e.g., working machines) filled to maximum capacity so long as spares are available for this purpose. The "interaction rule" governs the timing of replenishment. For all the models reviewed in this chapter, replenishment is assumed to be instantaneous upon failure of a machine.

The "solution" to a repairman model is usually thought of as being a specification of the probability distribution of the number of failed machines.

Repairman Models with Spares Provisioning

Limitations of the Literature

Only five papers were found which treated models of the type sought in the present investigation. Of these, only the two papers by Taylor and Jackson [232] and Toft and Boothroyd [240] presented original results for a model involving an arbitrary number of spares. The remaining three papers consist of a duplication of previous results by Wahi [244], an expected-value argument by George [68], and some highly imaginative and interesting work by Barlow [5] on the case of a single spare.

For the sake of completeness and for the convenience of potential users, the substance of the papers by Taylor and Jackson and Toft and Boothroyd will be reported in some detail. The present derivations make no attempt to adhere closely to the originals in either notation, starting point, or nature of development.

Negative-Exponential Occurrence Times

Taylor and Jackson [232], Toft and Boothroyd [240], Wahi [244], and Barlow [5] all have studied repairman problems with spares where machine service and failure times are negative exponential variates.

The topics of their papers somewhat indicate the widespread applicability of repairman models. Taylor and Jackson's model was devised to represent operation of fleet-operated aircraft engines at British Overseas Airways Corporation. Toft and Boothroyd's model was devised to represent colliery operations (where workable coal faces are subject to unforeseen closure due to geological faults, etc.). Wahi's duplication of the Taylor and Jackson model was presented in nonphysical terms; however, its application to maintenance of an analogue computer was demonstrated in an example.

A model may be constructed as follows. Suppose that there are c service channels (repairman) and a total of N + S machines of which S machines are spares. Let S_n denote the state of the system; λ_n, the (combined) rate at which machines are failing; and μ_n, the (combined) rate at which failed machines are being serviced when there are n ($n=0,1,2,\cdots,N+S$) failed machines in the system. Let $P_n(t)$ denote the probability that the system is in state S_n at time t and let p_n denote the corresponding steady-state probability of being in state S_n.

In order to completely specify the λ_n and μ_n, some additional assumptions are needed. Each of the cited investigators [5,232,240,244] has supposed that when a machine fails, it is (1) instantly placed into the queue of units awaiting service and (2) instantly replaced in operation by a spare if one is available. Now, note that when S + 1 machines are in the service facility, it is no longer possible to operate the system at the maximum (and desired) capacity of N operative machines. There are two possible cases:

Case I. All machines stop operating until servicing can restore the operational number of machines to N.

Case II. A number of machines less than N continues to operate.

Case I was first studied by Taylor and Jackson [232], 1954, under the added assumption that $S \geq c$.* For this case,

$$\left.\begin{array}{lll} \lambda_n = N\lambda, & \mu_n = n\mu, & n=0,1,2,\cdots,c; \\ \lambda_n = N\lambda, & \mu_n = c\mu, & n=c,c+1,c+2,\cdots,S; \\ \lambda_{S+1} = 0, & \mu_{S+1} = c\mu, & n=S+1; \end{array}\right\} \quad (40)$$

where λ is the failure rate of an individual machine and μ is the service rate of an individual channel.

The system can be shown to be a special instance of the general birth-and-death model of stochastic processes. The time-dependent equations of state follow from substituting Equations (40) into Equations (6 - 7). These equations were stated by Taylor and Jackson as the starting point in their analytic development.** It was noted that the equations were similar to those given by Feller [52] and, while the reference was not more specific, it appears that they are an adaptation of Feller's presentation ([52], p.416-421) of Palm's model [173] for the analogous problem without spares provisioning.

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*If $S<c$, it is obvious that $c-S-1$ of the channels need never be used and, accordingly, can be excluded from the model.

**The present notation differs from that of Taylor and Jackson; however, the equations cited here and those given by Taylor and Jackson ([232], p.98) are identical in mathematical content.

Taylor and Jackson next reduced the dynamic equations to a stationary set by taking limits as $t \to \infty$ (also obtainable by substituting Equations (40) into (9)) and solved these recursively to find

$$\left.\begin{aligned} p_n &= \left(\frac{N\lambda}{\mu}\right)^n \frac{1}{n!}\, p_o, && n=1,2,3,\cdots,c; \\ p_n &= \left(\frac{N\lambda}{\mu}\right)^n \frac{1}{c!c^{n-c}}\, p_o, && n=c,c+1,c+2,\cdots,S+1; \end{aligned}\right\} \tag{41}$$

where p_o is evaluated from the condition

$$\sum_{n=0}^{S+1} p_n = 1. \tag{42}$$

These results are just what is obtained by substituting (40) into (10-11).

The remainder of Taylor and Jackson's discussion was based on this steady-state solution. The statistics of interest (as functions of S) were identified as:

(i) the chance of the system breaking down, p_{S+1};

(ii) the operational efficiency of the system,

$$E_S = 100(1-p_{S+1})\%; \tag{43}$$

(iii) the average number of unserviceable machines,

$$U_S = \sum_{n=o}^{S+1} np_n; \tag{44}$$

(iv) the average number of machines waiting for service,

$$W_S = \sum_{n=c+1}^{S+1} (n-c)p_n; \tag{45}$$

(v) the average number of machines being serviced,

$$M_S = \sum_{n=0}^{c-1} np_n + c \sum_{n=c}^{S+1} p_n; \text{ and} \tag{46}$$

(vi) the service utilization factor,

$$F_S = \frac{100\ M_S}{c} \%. \tag{47}$$

However, these expressions were not evaluated for the given model. Instead, Taylor and Jackson sought to establish bounds on the behavior of the system with respect to variations in the number of spares provided.

Taylor and Jackson discussed the special case $N\lambda = c\mu$, showing that

$$p_o<p_1<p_2<\cdots<p_{c-1} = p_c = p_{c+1} = \cdots = p_{S+1}, \tag{48}$$

$$\frac{G}{S-c+1} < p_c < \frac{1}{S-c+2}, \quad \text{and} \tag{49}$$

$$\frac{G(S-c+2)}{2} < W_S < \frac{(S-c+1)}{2}; \tag{50}$$

where

$$G = \left(\frac{c}{c!}\right)\left[\sum_{n=0}^{c} \frac{c^n}{n!}\right]^{-1} < 1. \tag{51}$$

It follows that with increasing S, the rate of decrease of p_{S+1} is on the order of S^{-2} and the rate of increase of W_S is a constant between G/2 and 1/2. Thus, increasing the number of spare machines rapidly ceases to have material effect on reducing the probability of system failure and yields an increasing average number of machines waiting for repair. From this and other observations, Taylor and Jackson concluded that the special case $N\lambda = c\mu$ was likely to be uneconomical as in the classical cases $S=0$ and $S\to\infty$ (for which the model reduces to well-known classical queueing models), even with spares provisioning.

Taylor and Jackson next investigated "the only practical alternative," the case $N\lambda < c\mu$. Writing $x = N\lambda/(c\mu)$, they showed that

$$p_n < \frac{(1-x)x^{n-c}}{1 - x^{S-c+2}}, \quad n=c,c+1,c+2,\cdots,S+1; \text{ and} \tag{52}$$

$$W_S < \frac{x}{1 - x}; \tag{53}$$

from which it follows that p_n is a decreasing function of S and n, and that W_S has an upper bound independent of S. Hence, in contrast to the previous case, increasing the number of spare machines does have a beneficial effect in that the operational efficiency is increased but the average number of machines awaiting service is *not* increased.

In 1961, Wahi [244] presented a paper that was remarkably similar in format and results to that of Taylor and Jackson, although his cited references indicate that the work was independently based on Feller's birth-and-death theory approach [52]. Although no new contributions to theory were made, Wahi did give a helpfully explicit description of spares optimization for a variable-cost relation of the form

$$C(S) = C_1 p_{S+1} + C_2 S. \tag{54}$$

If $\hat{S}$ is the value of S at which C(S) has an optimum, then it is known (e.g., [200], p.295) that C(S) must satisfy

$$\Delta C(\hat{S}-1) < 0 < \Delta C(\hat{S}). \tag{55}$$

Using this condition, Wahi easily obtained the inequality,

$$p_{S+1}\Big|_{S=\hat{S}} - p_{S+1}\Big|_{S=\hat{S}+1} \le \frac{C_2}{C_1} \le p_{S+1}\Big|_{S=\hat{S}-1} - p_{S+1}\Big|_{S=\hat{S}}. \tag{56}$$

Wahi made the following observation which is encouraging to potential applications

> The results of this model can be useful for analyzing a wide range of industrial problems. The method of solution can be programmed easily for a computer to generate a set of tables which provides the operational characteristics of a system and also the optimum number of spares or service channels or both. These tables save complex manual computation procedure and will act as a quick guide to obtain knowledge of a system in terms of its operational characteristics. ([244], p.115.)

Attention is now directed toward Case II, where after exhaustion of available spares, the system continues to operate at reduced capacity. When $S \geq c$,

$$\left.\begin{array}{lll} \lambda_n = N\lambda, & \mu_n = n\mu, & n=0,1,2,\cdots,c; \\ \lambda_n = N\lambda, & \mu_n = c\mu, & n=c,c+1,c+2,\cdots,S; \\ \lambda_n = (N+S-n)\lambda, & \mu_n = c\mu, & n=S,S+1,S+2,\cdots,N+S; \end{array}\right\} \quad (57)$$

and when $S<c$,

$$\left.\begin{array}{lll} \lambda_n = N\lambda, & \mu_n = n\mu, & n=0,1,2,\cdots,S; \\ \lambda_n = (N+S-n)\lambda, & \mu_n = n\mu, & n=S,S+1,S+2,\cdots,c; \\ \lambda_n = (N+S-n)\lambda, & \mu_n = c\mu, & n=c,c+1,c+2,\cdots,N+S. \end{array}\right\} \quad (58)$$

Substituting these values into the general birth and death equations (6), one obtains the time-dependent equations of state which Toft and Boothroyd [240], 1959, used as the starting point in their discussion of colliery operations. The steady-state equations follow similarly from substituting Equations (57-58) into (9).

The steady-state solution is, for $S \geq c$,

$$\left.\begin{array}{ll} p_n = (N^n \rho^n p_o)/n!, & n=1,2,3,\cdots,c; \\ p_n = (N^n \rho^n p_o)/(c!c^{n-c}), & n=c,c+1,c+2,\cdots,S; \\ p_n = (N!N^S \rho^n p_o)/[(N+S-n)!c!c^{n-c}], & n=S,S+1,S+2,\cdots,N+S; \end{array}\right\} \quad (59)$$

and for $S<c$,

$$\left.\begin{aligned} p_n &= (N^n \rho^n p_o)/n!, && n=1,2,3,\cdots,S; \\ p_n &= (N!N^S \rho^n p_o)/[n!(N+S-n)!], && n=S,S+1,S+2,\cdots,c; \\ p_n &= (N!N^S \rho^n p_o)/[c!c^{n-c}(N+S-n)!], && n=c,c+1,c+2,\cdots,N+S; \end{aligned}\right\} \quad (60)$$

where in both cases $\rho = \lambda/\mu$, and p_o is to be evaluated from the condition

$$\sum_{n=0}^{N+S} p_n = 1. \quad (61)$$

Equations (59-61) are in the form obtained by Toft and Boothroyd.

The statistics of interest in the colliery problem were described by Toft and Boothroyd as being the average number of spare faces held at one time,

$$J = \sum_{n=0}^{S} (S-n)p_n, \quad (62)$$

and the average proportion of working time lost (i.e., when $n>S$),

$$L = \frac{1}{N} \sum_{n=S+1}^{N+S} (n-S)p_n. \quad (63)$$

No explicit formulae for these quantities were developed for their model; however, the authors presented several graphs which related the average proportion of working time lost to various values of ρ, c, and S.

Barlow [5],* 1962, in an expository paper, sought to exploit the relationship between repairman problems and the more general realm of queueing problems. His interest was in finding the distribution of the time to occurrence of a total failure (when all machines are being repaired or awaiting repair), its moments, and certain related quantities. To treat the case of negative-exponentially distributed machine failure and repair times, he introduced the "orthogonal-polynomial approach" to birth-and-death processes as advanced by Karlin and McGregor [118-121,123]** and Harris [79]. This approach permits one to calculate the time-dependent transition probabilities*** for the birth-and-death process. Since the method is designed to apply to any such process for which the λ_n and μ_n are specified, it can be applied without modification to any so-specified repairman model.

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*The portions of Barlow's paper that treat repairman problems without spares were described earlier in Chapter II under the heading of "devoted repairmen." Barlow's paper is typographically confusing in references to authors and equations. A more readable but less thoroughly documented version appears in the text by Barlow and Proschan ([6], pp.139-151).

**Karlin and McGregor have published other works [116,117,122,124] that are relevant in the present context.

***Recall that, as with any Markov process, summation of the transition probabilities $P_{ij}(t)$ over the initial states i yields the unconditional probabilities $P_j(t)$ of being in state j at time t. Hence, it appears that Barlow's exposition might offer a method for obtaining time-dependent solutions to the problems discussed in the preceding paragraphs.

Barlow demonstrated the approach on an example which, in the present terminology, would be called a Case II problem with $N = 1$, $S = 1$, and $c = 1$. Generally, if $S > c$, there will be $N + S + 1$ polynomials in the set, one of which is of degree $N + S$. This casts some serious doubts on the usefulness of the approach for large values of N and S.

> As we have seen for the exponential case, most questions of interest can be answered once we have obtained the associated orthogonal polynomials. Determining the roots of certain of these polynomials constitutes the principle difficulty. ([5], p.25; [6], p.146.)

Constant Service Times

In a short note, George [68], 1964, has described a scheme by which one can calculate the average number of operating machines in a system involving one repairman; $N + S$ machines of which S are spares; a negative-exponential time-to-failure variate for each machine; and constant service time T for each machine.

More General Models

Barlow [5] also discussed repairman problems with spares provisioning and involving more general distributions than the negative exponential. Two specific models were presented.

The first of these was a model possessing c service channels and c machines, of which $c - 1$ machines were spares. The machine failure times were assumed to be negative-exponentially distributed with mean time $1/\lambda$ and the service times were assumed to follow a general distribution with mean time β. Barlow identified this model with the telephone trunking problem of Poisson input to a finite number of channels with blocked calls cleared and quoted the solution from Sevast'yanov [202] as

$$p_n = \frac{(\lambda\beta)^n}{n!}\left[\sum_{j=0}^{c}\frac{(\lambda\beta)^j}{j!}\right]^{-1}, \quad n=0,1,2,\cdots,c, \tag{64}$$

where p_n is the steady-state probability that there are n failed machines in the system.

The second model also possessed c service channels and c machines of which c - 1 machines were spares. However, the machine failure times were assumed to follow a general distribution with mean α and the service times were assumed to be negative-exponentially distributed with mean $1/\mu$. Barlow stated that this model yields the same process as obtained in Takács' model [219] for another repairman problem (without spares). Takács' model (31-32) was presented in Chapter II. To adapt it to the present problem, substitute α, μ, and c in place of β, λ, and m, respectively, and redefine p_n to be the steady-state probability that there are n failed machines in the system.

It is apparent that these two models are *very* special cases and that an opportunity for their use will rarely arise in practical applications. However, such is the state of the art. It is clear that there is a need for more work in this area. Barlow's paper did indicate one direction this work might take.

Barlow suggested that the technique of the imbedded semi-Markov process could be used to solve more general models such as the following:

(1) S + 1 machines total, S spares, c service channels, general failure times, and negative-exponential service times; and

(2) S + N machines total, S spares, one service channel, negative-exponential failure times, and general service times.

An approach was indicated but not pursued to formulation or solution. It was suggested that portions of the solution could be obtained from Takács' complicated paper [224] on a telephone trunking problem.

Reliability-Oriented Investigations of the Repairman Model with Spares

A variation of the repairman problem with spares has been the subject of reliability-oriented investigations. The situation of interest is that where the system ceases to operate after the supply of spares has been exhausted (i.e., Case I of the previous section); where both the spares and the operating units are subject to time-dependent stochastic failure; where service time on a unit is independent of the mode of failure; and where, in some versions, both the spares and the operating units are monitored for failures (i.e., failures are not immediately detected).

An illuminating example of this type of analysis has been provided by Ashar [2], 1960. His problem was defined essentially like that of Taylor and Jackson [232] except that it contained the added provision that units in the spares inventory were subject to failure according to a negative-exponential distribution with parameter $\gamma\lambda$, where λ is the failure rate of an operating unit and γ is a real constant such that $0 \leq \gamma \leq 1$. Thus, in contrast to Equations (40) for Taylor and Jackson's model, the process may be characterized as follows for $c \leq S$:

$$\left.\begin{aligned} \lambda_n &= N\lambda + (S-n)\gamma\lambda, & \mu_n &= n\mu, & n&=0,1,2,\cdots,c; \\ \lambda_n &= N\lambda + (S-n)\gamma\lambda, & \mu_n &= c\mu, & n&=c,c+1,c+2,\cdots,S; \\ \lambda_{S+1} &= 0, & \mu_{S+1} &= c\mu, & n&=S+1. \end{aligned}\right\} \tag{65}$$

Actually, Ashar treated a somewhat different problem with $c = S + 1$. The above equations are provided only to show the relationship between Ashar's variation and a familiar repairman problem with spares.

Ashar developed the dynamic equations for his system using the usual argument about possible state transitions in a small interval of time Δt. He next transferred these equations into the Laplace transform domain and established a scheme for computing (only) $P^e_{S+1}(\tau)$, the unilateral Laplace transform of $P_{S+1}(t)$. He then showed how the reliability and the failure density functions of the system could be calculated from $P^e_{S+1}(\tau)$. Finally, Ashar developed a summation expression for the expected time to system failure.

From the viewpoint of the present investigation, Ashar's results are incomplete since they fail to fulfill the objective of modeling the state-dependent stochastic nature of the process. This is a common failing of the reliability-oriented investigations of repairman problems with spares and, from the present standpoint, the reports on such investigations must be regarded as a source of suggestions for possible variations on the basic models rather than as useful background material.

To provide an entrée into this area of the literature, reference is made to the papers by Ashar [2], Buzzanca and Goldstein [23], Natarajan [165], Ōmae [170,171], Rohn [191], and Srinivasan [205].

Networks of Queues

Networks, Cyclic Queues, and Repairman Models

This section will present an abbreviated discussion of queueing network literature as a preface to the discussion of cyclic queues which follows. As will be observed, cyclic queues form a subset of queueing networks. More specifically, a cyclic queueing network is a tandem queueing network with total feedback from the last to the first stage. Cyclic queues are themselves of interest because they are a generalization of both regular repairman models and those with spares. The existence of this hierarchy of classes of queueing models,

$$\begin{aligned}&\{\text{Repairman Models}\} \subset \{\text{Repairman Models with Spares}\}\\ &\quad\subset \{\text{Cyclic Queues}\} \subset \{\text{Networks of Queues}\},\end{aligned} \tag{66}$$

has important implications toward the techniques potentially available for the study of repairman models with spares.

There are two usual approaches to the study of networks of queues. The first consists of a reticulation of the network into discrete stages, each comprised of a service station and its associated waiting line as shown in Figure 2. Such isolation of queues becomes useful if it can be shown that the steady-state departure process from a queueing system can be expressed in a form independent of the state parameters of that system. Unfortunately, this is not generally the case unless the network is comprised of M|M|C queues. The second approach consists of analyzing several component queues as a composite group of queues in parallel (Figure 5), tandem (Figure 6), or feedback (Figure 7) arrays. The key element in either approach--but especially in the first--is to obtain a representation of the departure process.

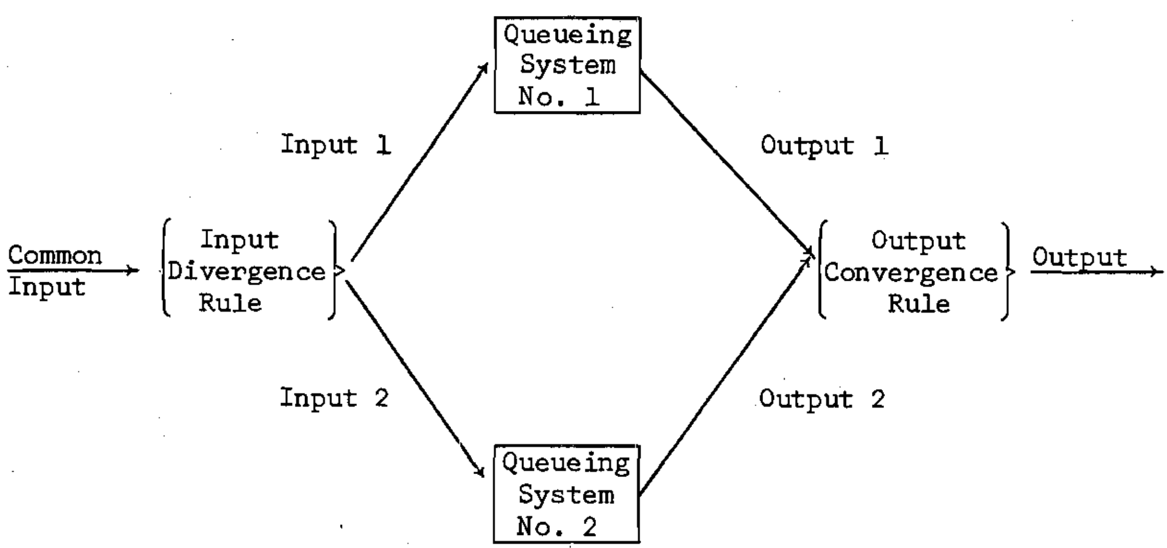

Figure 5. Two Queues in Parallel

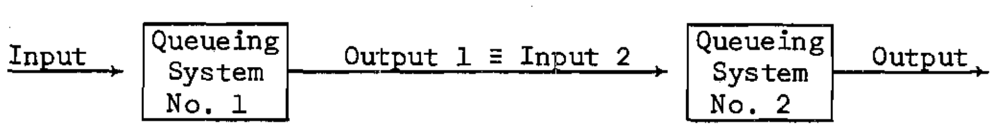

Figure 6. Two Queues in Series ("Tandem" Queues)

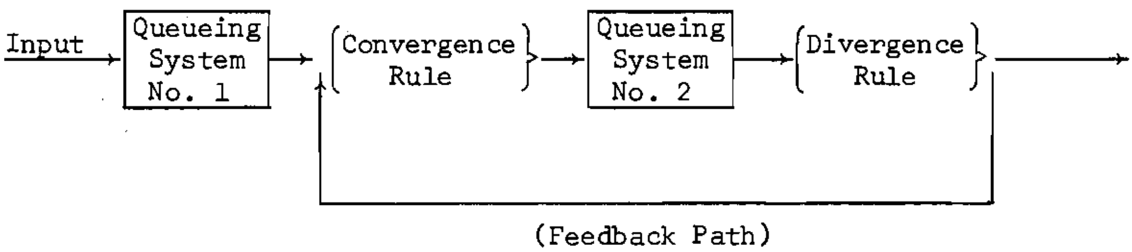

Figure 7. Example of Feedback in a Network of Queues

Departure Processes

Only steady-state results have been obtained for the departure process from a single queueing system. The first complete derivation of an output distribution was given in 1956 by Burke [21], who showed that the output of the M|M|c queue is itself Poisson with the same parameter as the input. Independent derivations of and/or partial converses to this fact have been given by Reich [184], Finch [56,58], Chesbrough [26], Cohen [30], and Jenkins [108]. Only one generalization of Burke's result is known: Mirasol [153] has shown that the output process of the M|G|∞ queue is Poisson. Kendall [130] has very simply obtained Mirasol's result for the special case where the service time distribution is a step function.

Methods for obtaining Laplace transforms of more general queue output distributions have been described by Chang [24] and Chesbrough [26]. The correlation structure of departure processes have been studied by Jenkins [108], for $M|E_k|1$ queues, and Cox [39], for gamma-distributed service times. For survey material on departure processes, reference is made to Chesbrough [26] and Reich [186].

Networks of Queues

Queues in Tandem. The earliest known result on tandem queues is Erlang's proof [18] that k $M|M|1$ queues in tandem with no inter-stage waiting (i.e., phase-type servicing) are equivalent to a single $M|E_k|1$ queue.* This result and Burke's $M|M|c$ output result [21] have proved to be the underlying influences on the study of queues in series. Among the more significant papers on tandem queues are those by Burke [22], De Baun and Katz [43], Ghosal [69], Hunt [94], J. R. Jackson [95-97], R. R. P. Jackson [98,99], Loynes [146], Morse [159], Nelson [166,167], O'Brien [168], Reich [184,185], and Sacks [199]. A number of interesting applications have been studied by Morse [159] and Morris [158]. Incidentally, the paper by R. R. P. Jackson [99] was directly motivated by observation of the internal operation of an aircraft engine overhaul facility.

Additional references may be obtained from the bibliographies of the cited mainstream papers and from the surveys given by Lee [144], Saaty ([196], Ch. 12; [197]), Chesbrough [26], and Reich [186].

Queues in Parallel. Compared with the fairly extensive literature on tandem queues, that on parallel queues is quite limited. Of course, it is possible to model some of the simpler parallel queueing situations with the standard (homogeneous servers) multichannel models of queueing theory. Multichannel models with heterogenous servers (usually modified $M|M|c$) have been studied by, e.g., Daru [42], Gani

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*This result has important implications for the models developed in Chapter V.

[63], Gani and Pyke [64],* Gumbel [75], and Krishnamoorthi [141] for various assumptions about the queue discipline for a common waiting line. Similar studies have been made by, e.g., Haight [77], Wilkins [249], and Kingman [134] for the case of distinct waiting lines.

Using the above-mentioned results due to Burke [21] and Erlang [18] and the additional fact that the sum of Poisson variates is itself a Poisson variate (see, e.g., Romani [192]), Cohen [30] and J. R. Jackson [95] have studied networks of $M|M|c$ queues in various parallel and series configurations. Nelson's simulation experiments [167] should also be mentioned in this connection. Reference is made to Morse [159] and Morris [158] for interesting applications and to Chesbrough [26] and Saaty ([196], Ch. 12; [197]) for additional references and survey material.

Queues with Feedback. Repairman models, repairman models with spares, and cyclic queueing models have been discussed in some detail elsewhere in this survey. All such models are examples of queueing with feedback. Attention is also directed to the work of Takács [229] and Chang [25] on models of single-server queues with feedback. Takács' model is basically that of a $M|G|1$ queue except that departing units are assumed to immediately rejoin the queue with probability p ($0<p<1$). Chang's models are adaptations of Takács' model which involve the additional ramifications of arrivals from N priority classes and

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*The papers [63,64] were a treatment of an equivalent problem in dam theory. For a thorough discussion of basic dam theory, see Gani [62]. For a unifying discussion of queues and dams, see Prabhu [183].

servicing on a time-sharing basis.*

Cyclic Queues**

Koenigsberg's Work

Cyclic queues are an innovation due to E. Koenigsberg [138,139]. Their relevance in the present discussion was described in the introduction to the previous section and illustrated in Equation (66). Cyclic queues are useful in the study of maintenance flow systems in which machine operation, repair, or both occur in several discrete phases. The simplest sets of cyclic queues are the repairman models and the repairman models with spares.

A description of cyclic queueing systems can perhaps best be given in relation to Koenigsberg's two pioneering papers. Koenigsberg [138], 1958, modeled a coal-cutting problem as a set of k tandem $M|M|1$ queues (operations performed) serving N units (mine faces) in rotation. Each operation had a negative exponential service time distribution and, after units 1 through N were served, the sequence was repeated. This type of closed system he called "cyclic queues." (See Figure 8.) Koenigsberg obtained a multivariate, steady-state probability distribution for the number of units in each waiting line.

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*Chang's models [25] are applicable to the analysis of time-sharing computer (and other) systems. Studies in this area of application have profited from and made contributions to the theory of queues with feedback. See Chang's bibliography for an entrée into this literature.

**Additional discussion of results from cyclic queueing theory appears in Chapter V.

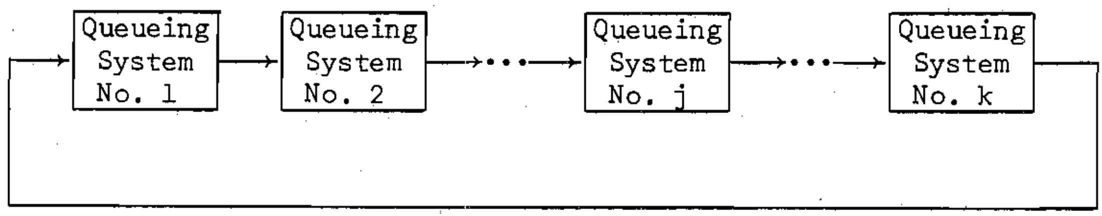

Figure 8. Cyclic Queues

In a second paper [139], Koenigsberg illustrated the relationship between cyclic queues and the subset of repairman models, including those with spares. His model two $M|M|c$ stages visited in rotation by N machines. The first stage was regarded as being a repair facility with c_1 servers and the second as a working area with a capacity of c_2 working positions for machines. The waiting line associated with the first stage contained failed machines awaiting repair and that associated with the second stage contained repaired, but idle, machines awaiting a working position. For $c_2 < N$, it can be seen that this system is equivalent to that studied in another way by Toft and Boothroyd [240], that is, a repairman model with spares. For $c_2 = N$, the system reduces to the simple repairman model studied by Palm [173] and others.

As in the previous paper, Koenigsberg found (bivariate) steady-state probability distributions for the number of units in each waiting line. Formulae were given for a number of statistics of interest and several numerical examples were used to illustrate their application to systems design.

Koenigsberg's work has proved useful in a number of applications. For example, Hall [78] has applied it to a system maintainability study, Leese [145] has used it in an aircraft maintenance study, and Mirasol [154] has applied it to the investigation of a large logistics system. Fujimoto and Oda [61] have applied the theory of cyclic queues (source not cited) to the study of the circulation of spare parts for the rolling stock of a railroad.

Generalizations of Koenigsberg's Models

Finch [57], 1959, studied the same problem as Koenigsberg [138]. However, in his model, a unit returned to the j*th* queue with probability p_j upon completing service at the last queue. Finch obtained the multivariate, steady-state probability distribution for the number of units in each waiting line.

Benson and Gregory [14], 1961, generalized Koenigsberg's model in several directions. Their model also involved a set of k tandem $M|M|1$ queues in a closed loop; however, two refinements were added. First, it was assumed that a unit departing from stage j would require a finite "transit" time to reach the next stage. All transit times were assumed to be negative exponential variates. Second, it was assumed that, in addition to the closed-loop units, each stage also received arrivals from outside the system. The exogenous arrivals were assumed to join the queue before a stage, receive nonpreferential FIFO service, and afterwards depart to an external location. All exogenous interarrival time distributins were assumed to be negative exponential. Multivariate, steady-state probability distributions for the number of units in each waiting line were reported and used in formulae for various statistics of interest.

More recent work by Posner and Bernholtz [179,180], 1967, has resulted in several generalizations of Benson and Gregory's model (excluding exogenous arrivals). The transit times of a unit from stage j to stage j+1 were assumed to be distributed according to a general function $G_j(\cdot)$. The service time of stage j was assumed to be a negative-exponentially distributed variate, with parameter λ_{jn} dependent upon the number n_j of units in the waiting line before that stage. Using the supplementary variable technique, the authors obtained multivariate, steady-state probabilities for the states of the system and discussed the usual statistics of interest. Their first paper [180] treated a two-stage model and their second paper [179] extended analysis to the more general model with k stages.

In other recent work, Gordon and Newell [71] have studied Koenigsberg's model [138] with the additional assumption of restricted waiting space between stages. Stationary, multivariate state probabilities were obtained for three cases: (1) a two-stage network, (2) a k-stage network with so few units that blocking is negligible, and (3) a k-stage network with so many units that blocking predominates.

Gordon and Newell [70], Pelczynski [177], Posner and Bernholtz [179,181,182], and Swersey [210,211] all have studied generalizations of cyclic queues in which units completing service at the *ith* stage join the queue at the *jth* stage $(i,j=1,2,3,\cdots,k)$ with transition probability p_{ij}. These finite models, often referred to as "closed queues," are not restricted to cyclic queueing situations but apply in general to complex networks of queues.

CHAPTER IV

POISSON REPAIRMAN MODELS WITH SPARES

Introduction

This chapter will present a variety of new results and observations on repairman problems with spares in which the units have negative-exponentially distributed failure and service times. Topics to be discussed are (in order) spares which fail in storage, artificially extended service intervals, reduction of system failures, the prospects for time-dependent solutions, repairmen with ancillary duties, faulty repairs, cyclic queues, transit delays, and the servicing of aircraft engines. The principal methods of attack are those of Poisson queueing theory (see aspects outlined in Chapter I and the general survey by Saaty [196]) and those of cyclic and finite queueing network theory (see survey in Chapter III).

Previous work on Poisson repairman problems with spares has been done by Taylor and Jackson [232], by Toft and Boothroyd [240], and, in the context of cyclic queueing problems, by Koenigsberg [138-139]. Their papers are reviewed in Chapter III. Many of the new models are characterizable as being extensions, embellishments, or generalizations of the Taylor and Jackson model (Equations (41-42)) and/or Toft and Boothroyd model (Equations (59-61)); hence, the reader may find it useful to review the assumptions and relations underlying their models.* Further,

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*The discussion of traffic intensity centered around Equations the reader may find the material of Chapter II, on repairman models without spares, interesting as general background information. He will find that some of the new models are analogues to these regular repairman models while some provide new regular repairman models when the spares parameter is set to zero.

The emphasis in this chapter has been on the formulation of new analytic representations, involving symbolically general parameters, for a variety of repairman systems with spares. Because of the complexity of the equations obtained, no effort was made to write explicit formulae for the statistics of possible interest. It was found in most instances that the writing of such formulae would consist essentially of a re-listing of definitions with no corresponding reduction in the complexity of the individual terms of a summation. Taylor and Jackson and Toft and Boothroyd encountered the same difficulty for even their relatively simple models. Accordingly, we follow their lead and refer the user to the standard definitions listed in Chapter III (Equations (43-47,62-63) with obvious choices for the summation bounds), from which he should be able to easily calculate these statistics for specific cases of interest to him.

Models with Spares which Fail in Storage

Ashar's Notion

In Chapter III, a brief description was given of Ashar's notion

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(48-53) will be of particular interest. The conclusion that the only "practical" case is that for which the traffic intensity is less than unity (i.e., $N\lambda/c\mu < 1$) will be seen to apply to all the models of Chapter IV.

of spares which can fail during storage. Specifically, Ashar [2] assumed that units in the spares inventory were subject to failure according to a negative-exponential distribution with mean rate $\gamma\lambda$, where λ is the failure rate of an operating machine and γ is a real constant such that $0 \le \gamma \le 1$. This might be the case where standby units are maintained in a semi-operative state. It was assumed that failures of any type were immediately detected and entered into the repair queue. As previously noted, Ashar did not fully develop a model of his problem but instead confined his investigation to a study of the probability of a total system failure for a given operating policy.

In this section, several repairman models with spares will be developed in which the spares are subject to failure along the lines of Ashar's notion. In general, suppose that there are c service channels (repairmen) and a total of N+S machines of which S machines are spares. Adopting the notation of the previously described birth and death model, Equations (6-11), we interpret λ_n as the (combined) rate at which machines are failing and μ_n as the (combined) rate at which failed machines are being serviced when there are n $(n=0,1,2,\cdots,N+S)$ failed machines in the system. Assume that the failure time distribution for an operating machine and the service time distribution of a failed machine are negative exponential with mean rates λ and μ, respectively.

Note that when S+1 machines are in the service facility, it is no longer possible to operate the system at the maximum capacity of N operative machines. There are accordingly two cases of interest:

Case I. All machines stop operating until servicing can restore the operational number of machines to N.

Case II. A number of machines less than N continues to operate.

Case I

For Case I, we make the natural assumption that $S \geq c$.* The model sought is a special case of the general birth-and-death model. It is completely defined by a specification of the λ_n and μ_n as follows:

$$\left.\begin{array}{lll} \lambda_n = [N+\gamma(S-n)]\lambda, & \mu_n = n\mu, & n=0,1,2,\cdots,c; \\ \lambda_n = [N+\gamma(S-n)]\lambda, & \mu_n = c\mu, & n=c,c+1,c+2,\cdots,S; \\ \lambda_{S+1} = 0, & \mu_{S+1} = c\mu, & n=S+1. \end{array}\right\} \quad (67)$$

The time-dependent and steady-state equations of the model may be obtained by substituting these values into Equations (6-7) and (9), respectively.

The stationary solution is obtained from Equations (10-11):

$$\left.\begin{array}{ll} p_n = \rho^n B_n\, p_o/n!, & n=1,2,3,\cdots,c; \\ p_n = \rho^n B_n\, p_o/(c!c^{n-c}), & n=c,c+1,c+2,\cdots,S+1; \end{array}\right\} \quad (68)$$

where

$$\rho = \lambda/\mu; \quad (69)$$

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*Under the alternative assumption, $S<c$, it is clear that at *any* time there are at least $c-S$ unused service positions.

$$B_n = \prod_{j=0}^{n-1} [N+\gamma(S-j)], \qquad n=1,2,3,\cdots,S+1; \tag{70}$$

and p_o is to be calculated from the certainty condition for a regular solution,

$$\sum_{n=0}^{S+1} p_n = 1. \tag{71}$$

Simpler solutions can be obtained for two subcases. When $\gamma = 0$, there are no failures of spares during storage and thus Equations (68) reduce to the form (41) obtained by Taylor and Jackson [232]. When $\gamma = 1$, stored spares fail at the same rate as operating units. This might be the case in, say, military field operations where an inhospitable climate is the primary cause of failures. With $\gamma = 1$ we have

$$B_n\Big|_{\gamma=1} = \prod_{j=0}^{n-1} [N+S-j] = \frac{(N+S)!}{(N+S-n)!}, \tag{72}$$

so that

$$\left.\begin{aligned} p_n &= \binom{N+S}{n}\rho^n p_o, && n=1,2,3,\cdots,c; \\ p_n &= \binom{N+S}{n}\rho^n p_o[n!/(c!c^{n-c})], && n=c,c+1,c+2,\cdots,S+1; \end{aligned}\right\} \tag{73}$$

where again p_o is to be evaluated using Equation (71). If $N = 1$ in Equations (73), the solution reduces to Palm's result [173] for the case of c repairmen, $S + 1$ machines, and no spares (cf. Equations (17)).

Case II with $S \geq c$

For Case II with $S \geq c$, the λ_n and μ_n are

$$\left.\begin{array}{lll} \lambda_n = [N+\gamma(S-n)]\lambda, & \mu_n = n\mu, & n=0,1,2,\cdots,c; \\ \lambda_n = [N+\gamma(S-n)]\lambda, & \mu_n = c\mu, & n=c,c+1,c+2,\cdots,S; \\ \lambda_n = (N+S-n)\lambda, & \mu_n = c\mu, & n=S,S+1,S+2,\cdots,N+S. \end{array}\right\} \quad (74)$$

The model's time-dependent and steady-state equations are obtained by substituting these λ_n and μ_n into Equations (6-7) and (9), respectively. Solving the latter recursively, or using Equations (10-11), we find the stationary solution

$$\left.\begin{array}{ll} p_n = \rho^n B_n p_o/n!, & n=1,2,3,\cdots,c; \\ p_n = \rho^n B_n p_o/(c!c^{n-c}), & n=c,c+1,c+2,\cdots,S; \\ p_n = N!\rho^n B_S p_o/[(N+S-n)!c!c^{n-c}], & n=S,S+1,S+2,\cdots,N+S; \end{array}\right\} \quad (75)$$

where ρ and B_n are defined by Equations (69) and (70), respectively. and where p_o is to be evaluated using the normalizing condition

$$\sum_{n=0}^{N+S} p_n = 1. \quad (76)$$

When $\gamma = 0$, Equations (75) reduce to the form (59) obtained by Toft and Boothroyd [240] for the case of spares which don't fail during storage. When $\gamma = 1$, we have the B_n as in Equations (72), so that Equations (75) reduce to the simpler form

$$\left.\begin{aligned} p_n &= \binom{N+S}{n}\rho^n p_o, && n=1,2,3,\cdots,c; \\ p_n &= \binom{N+S}{n}\rho^n p_o[n!/(c!c^{n-c})], && n=c,c+1,c+2,\cdots,N+S; \end{aligned}\right\} \quad (77)$$

where again p_o is to be evaluated from the condition (76).

<u>Case II with $S < c$</u>

For Case II with $S < c$, the λ_n and μ_n are

$$\left.\begin{aligned} \lambda_n &= [N+\gamma(S-n)]\lambda, & \mu_n &= n\mu, & n&=0,1,2,\cdots,S; \\ \lambda_n &= (N+S-n)\lambda, & \mu_n &= n\mu, & n&=S,S+1,S+2,\cdots,c; \\ \lambda_n &= (N+S-n)\lambda, & \mu_n &= c\mu, & n&=c,c+1,c+2,\cdots,N+S. \end{aligned}\right\} \quad (78)$$

As for $S \geq c$, the relevant equations of the model are obtained by substituting the λ_n and μ_n into Equations (6-11). The stationary solution is

$$p_n = \rho^n B_n p_o/n!, \quad n=1,2,3,\cdots,S;$$
$$\vdots$$

$$\left.\begin{aligned} p_n &= N!\rho^n B_S p_o/[(N+S-n)!n!], & n&=S,S+1,S+2,\cdots,c; \\ p_n &= N!\rho^n B_S p_o/[(N+S-n)!c!c^{n-c}], & n&=c,c+1,c+2,\cdots,N+S; \end{aligned}\right\} \quad (79)$$

where ρ, B_n, and p_o are given by Equations (69), (70), and (76) respectively.

When $\gamma = 0$, Equations (79) reduce to the form (59) obtained by Toft and Boothroyd [240]. When $\gamma = 1$, we have the B_n defined by Equations (72), so that Equations (79) reduce to the simpler form

$$\left.\begin{aligned} p_n &= \binom{N+S}{n}\rho^n p_o, & n&=1,2,3,\cdots,c; \\ p_n &= \binom{N+S}{n}\rho^n p_o[n!/(c!c^{n-c})], & n&=c,c+1,c+2,\cdots,N+S; \end{aligned}\right\} \quad (80)$$

where again p_o is to be evaluated from the condition (76). It will be observed that the p_n for $\gamma = 1$ are the same whether $S \geq c$ or $S < c$ (compare Equations (77) and (80)). This is to be expected since the arrival rate of failed machines at the service facility depends only upon the total number of nonfailed machines (N+S-n) and not where they are located in the system.

A Model with Extended Service Intervals

Schemes for Reducing the Number of System Failures

Consider Taylor and Jackson's model [232] for which the *proportions* p_n of total time spent in a state with n failed machines are given by Equations (41-42). Suppose it is desired to reduce the *number* of system breakdowns (when n=S+1) because of, say, exorbitant setup costs.

The duration of a system breakdown is the time required to change the state of the system from S + 1 to S, that is, the time required to repair one machine. Thus, the expected duration of a system breakdown is $1/(c\mu)$ and the average number $\bar{b}(T)$ of system breakdowns that occur during the interval $[0,T]$ is just $c\mu T p_{S+1}$, that is

$$\bar{b}(T) = c\mu T\left(\frac{N\lambda}{\mu}\right)^{S+1} \frac{1}{c!c^{S-c+1}}\left[\sum_{n=0}^{c-1} \frac{(N\lambda/\mu)^n}{n!} + \sum_{n=c}^{S+1} \frac{(N\lambda/\mu)^n}{c!c^{n-c}}\right]^{-1}, \quad (81)$$

where p_{S+1} was obtained from (41-42). Therefore, schemes for reducing the frequency of system breakdowns might involve any or all of the elements: expansion of the service center's multiservicing capability ($c\uparrow$); innovations to increase the service rate of a channel ($\mu\uparrow$), purchase of more reliable machines ($\lambda\downarrow$); purchase of more productive machines ($N\downarrow$); and purchase of more spares ($S\uparrow$). Any such schemes would necessarily involve a capital investment and accordingly would have to be cost justified in an engineering economy study.

We now derive a model to show the effect of using an alternative scheme that has the advantage of *not* requiring new capital investment. The scheme sacrifices available production time in order to reduce the frequency of system breakdowns/setups, so again implementation would have to be cost justified.

Description of the Model

In the Taylor and Jackson model, a system failure occurs when the number of failed machines exceeds the spares reserve. After a system breakdown (when $n=S+1$), the repairmen restore the number of functional machines to S and restart the system. As a result of this procedure, the system will return to state $S + 1$ with the next failure unless the repairmen have managed to complete another repair in the meantime.

Suppose now that the operating rules have been altered to allow the repairmen to provide a number of spares before restarting the system. Let x ($0 \le x < S$) be the number of nonfunctional machines in the system when it is restarted. Let p_n ($n=0,1,2,\cdots,S$) denote the stationary probability of there being n failed machines in the system while it is operating. Let q_n ($n=x+1,x+2,x+3,\cdots,S+1$) denote the stationary probability of there being n failed machines in the system during an "extended service interval" (that is, while the system is down). Let $P_n(t)$ and $Q_n(t)$ be the dynamic probabilities corresponding to p_n and q_n, respectively. It can be seen that these p_n have the same interpretation as those for Taylor and Jackson's model, while the sum $q_{x+1} + q_{x+2} + \cdots + q_{S+1}$ can be related to Taylor and Jackson's p_{S+1}. More specifically, we have included a number of supplementary states (for which the stationary probabilities are the q_n) in order to represent our process as a Markov chain.

Using the technique of considering possible happenings during a small interval of length Δt as $\Delta t \to 0$ (see Chapter I), or by appropriately modifying Taylor and Jackson's formulation [232], we obtain the dynamic equations,

$$\frac{dP_o(t)}{dt} = -N\lambda P_o(t) + \mu P_1(t) + \delta_{x,o}\mu Q_1(t); \tag{82}$$

$$\frac{dP_n(t)}{dt} = -(N\lambda+n\mu)P_n(t) + N\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

$$+ \delta_{x,n}(n+1)\mu Q_{n+1}(t), \qquad n=1,2,3,\cdots,c-1; \qquad (83)$$

$$\frac{dP_n(t)}{dt} = -(N\lambda+c\mu)P_n(t) + N\lambda P_{n-1}(t) + c\mu P_{n+1}(t) + \delta_{x,n}c\mu Q_{n+1}(t), \qquad (84)$$

$$n=c,c+1,c+2,\cdots,S-1;$$

$$\frac{dP_S(t)}{dt} = -(N\lambda+c\mu)P_S(t) + N\lambda P_{S-1}(t); \qquad (85)$$

$$\frac{dQ_{S+1}(t)}{dt} = -c\mu Q_{S+1}(t) + N\lambda P_S(t); \qquad (86)$$

which hold for $0 \le x < S$. These equations are to be supplemented with

$$\frac{dQ_n(t)}{dt} = -n\mu Q_n(t) + (n+1)\mu Q_{n+1}(t), \quad n=x+1,x+2,x+3,\cdots,c-1; \qquad (87)$$

$$\frac{dQ_n(t)}{dt} = -c\mu Q_n(t) + c\mu Q_{n+1}(t), \qquad n=c,c+1,c+2,\cdots,S \qquad (88)$$

if $x < c$; and supplemented with

$$\frac{dQ_n(t)}{dt} = -c\mu Q_n(t) + c\mu Q_{n+1}(t), \quad n=x+1,x+2,x+3,\cdots,S \qquad (89)$$

if $x \ge c$. The symbol $\delta_{x,n}$ is the Kronecker delta,

$$\left.\begin{array}{ll}\delta_{i,j} = 1, & \text{if } i=j; \\ \delta_{i,j} = 0, & \text{if } i\neq j.\end{array}\right\} \tag{90}$$

In each case (82 – 89), the subscript of $(d/dt)P_n(t)$ or $(d/dt)Q_n(t)$ shows the state for which the equation was written. The right-hand side of the equation relates the flows into and out of that state.

Steady-State Solution

The Case $S > x > c$. In the steady-state with $S > x > c$, Equations (82-86,89) become

$$0 = -N\lambda p_o + \mu p_1; \tag{91}$$

$$0 = -(N\lambda+n\mu)p_n + N\lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad n=1,2,3,\cdots,c-1; \tag{92}$$

$$0 = -(N\lambda+c\mu)p_n + N\lambda p_{n-1} + c\mu p_{n+1}, \quad n=c,c+1,c+2,\cdots,x-1; \tag{93}$$

$$0 = -(N\lambda+c\mu)p_x + N\lambda p_{x-1} + c\mu p_{x+1} + c\mu q_{x+1}; \tag{94}$$

$$0 = -(N\lambda+c\mu)p_n + N\lambda p_{n-1} + c\mu p_{n+1}, \quad n=x+1,x+2,x+3,\cdots,S-1; \tag{95}$$

$$0 = -(N\lambda+c\mu)p_S + N\lambda p_{S-1}; \tag{96}$$

$$0 = -c\mu q_n + c\mu q_{n+1}, \quad n=x+1,x+2,x+3,\cdots,S; \tag{97}$$

$$0 = -c\mu q_{S+1} + N\lambda p_S. \tag{98}$$

Equations (91-93) may be evaluated recursively to obtain

$$p_n = \left(\frac{N\lambda}{\mu}\right)^n \frac{1}{n!} p_o, \quad n=1,2,3,\cdots,c; \tag{99}$$

$$p_n = \left(\frac{N\lambda}{\mu}\right)^n \frac{1}{c!c^{n-c}} p_o, \quad n=c,c+1,c+2,\cdots,x. \tag{100}$$

Equation (95) may be recognized as being a second-order, linear, homogeneous difference equation with constant coefficients. It therefore has solutions of the form

$$p_n = A\beta^n, \tag{101}$$

where A is an arbitrary constant. Substituting (101) into (95) and eliminating common factors establishes the condition

$$0 = -(N\lambda+c\mu)\beta + N\lambda + c\mu\beta^2, \tag{102}$$

which, when solved for β, yields two roots,

$$\beta_1 = 1 \quad \text{and} \quad \beta_2 = \frac{N\lambda}{c\mu}. \tag{103}$$

The general solution to Equations (95) is thus of the form

$$p_n = A_1\left(\frac{N\lambda}{c\mu}\right)^n + A_2, \quad n=x,x+1,x+2,\cdots,S; \tag{104}$$

where A_1 and A_2 are arbitrary constants.

Two conditions are needed to evaluate A_1 and A_2. The first is provided by Equation (96), *viz.*,

$$0 = -(N\lambda+c\mu)[A_1\left(\frac{N\lambda}{c\mu}\right)^S + A_2] + N\lambda[A_1\left(\frac{N\lambda}{c\mu}\right)^{S-1} + A_2], \qquad (105)$$

or

$$A_2 = -A_1\left(\frac{N\lambda}{c\mu}\right)^{S+1}. \qquad (106)$$

Thus, Equation (104) becomes

$$p_n = A_1\left[\left(\frac{N\lambda}{c\mu}\right)^n - \left(\frac{N\lambda}{c\mu}\right)^{S+1}\right], \quad n=x,x+1,x+2,\cdots,S. \qquad (107)$$

A second condition is provided by the required continuity between the two Equations (100) and (107) (which, incidentally is reflected also in (94)). Equating these for n=x, we find

$$\left(\frac{N\lambda}{\mu}\right)^x \frac{1}{c!c^{x-c}} p_o = A_1\left[\left(\frac{N\lambda}{c\mu}\right)^x - \left(\frac{N\lambda}{c\mu}\right)^{S+1}\right], \qquad (108)$$

or

$$A_1 = p_o(c^c/c!)\left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-x+1}\right]^{-1}. \qquad (109)$$

Equation (107) now becomes

$$p_n = \left(\frac{N\lambda}{\mu}\right)^n \frac{1}{c!c^{n-c}} p_o \left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-n+1}\right] \left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-x+1}\right]^{-1}, \qquad (110)$$

$$n=x,x+1,x+2,\cdots,S.$$

To determine the q_n, we simply combine Equations (97) and (98) to obtain

$$q_n = \frac{N\lambda}{c\mu} p_S, \qquad n=x+1,x+2,x+3,\cdots,S+1. \qquad (111)$$

Thus,

$$q_n = \left(\frac{N\lambda}{\mu}\right)^{S+1} \frac{p_o}{c!c^{S-c+1}} \left[1 - \left(\frac{N\lambda}{c\mu}\right)\right] \left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-x+1}\right]^{-1}, \qquad (112)$$

$$n=x+1,x+2,x+3,\cdots,S+1.$$

We now define p_{S+1} as the total probability that the system is in the failed state. Then,

$$p_{S+1} = \sum_{n=x+1}^{S+1} q_n = \left(\frac{N\lambda}{\mu}\right)^{S+1} \frac{(S-x+1)}{c!c^{S-c+1}} p_o \left[1 - \left(\frac{N\lambda}{c\mu}\right)\right] \left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-x+1}\right]^{-1}. \qquad (113)$$

The complete solution is given by Equations (99,100,110,113). Grouping these results, we have for $S > x > c$,

$$\left.\begin{aligned}
p_n &= \frac{1}{n!}\left(\frac{N\lambda}{\mu}\right)^n p_o, && n=1,2,3,\cdots,c;\\
p_n &= \frac{1}{c!c^{n-c}}\left(\frac{N\lambda}{\mu}\right)^n p_o, && n=c,c+1,c+2,\cdots,x;\\
p_n &= \frac{1}{c!c^{n-c}}\left(\frac{N\lambda}{\mu}\right)^n p_o \left[1-\left(\frac{N\lambda}{c\mu}\right)^{S-n+1}\right]\left[1-\left(\frac{N\lambda}{c\mu}\right)^{S-x+1}\right]^{-1}, &&\\
& && n=x,x+1,x+2,\cdots,S;\\
p_{S+1} &= \frac{(S-x+1)}{c!c^{S-c+1}}\left(\frac{N\lambda}{\mu}\right)^{S+1} p_o \left[1-\left(\frac{N\lambda}{c\mu}\right)\right]\left[1-\left(\frac{N\lambda}{c\mu}\right)^{S-x+1}\right]^{-1}; &&
\end{aligned}\right\} \tag{114}$$

where p_o is to be evaluated from the normalizing condition

$$\sum_{n=0}^{S+1} p_n = 1. \tag{115}$$

The Case $x = S$. Until now the case $x = S$ has been excluded from the development. By making the substitution $x = S$ in Equations (114) and simplifying the result, we obtain Equations (41) which *are* the solution for the case $x = S$. Thus, Taylor and Jackson's model follows as a special case of the current model.

The Case $x = c$. In the steady-state, Equations (82-86,89) become

$$0 = -N\lambda p_o + \mu p_1; \tag{116}$$

$$0 = -(N\lambda+n\mu)p_n + N\lambda p_{n-1} + (n+1)\mu p_{n+1}, \qquad n=1,2,3,\cdots,c-1; \tag{117}$$

$$0 = -(N\lambda+c\mu)p_c + N\lambda p_{c-1} + c\mu p_{c+1} + c\mu q_{c+1}; \tag{118}$$

$$0 = -(N\lambda+c\mu)p_n + N\lambda p_{n-1} + c\mu p_{n+1}, \qquad n=c+1,c+2,c+3,\cdots,S-1; \tag{119}$$

$$0 = -(N\lambda+c\mu)p_S + N\lambda p_{S-1}; \tag{120}$$

$$0 = -c\mu q_n + c\mu q_{n+1}, \qquad n=c+1,c+2,c+3,\cdots,S; \tag{121}$$

$$0 = -c\mu q_{S+1} + N\lambda p_S. \tag{122}$$

It can be verified by substitution that Equations (116-122) are satisfied by the p_n and q_n of (112,114). Therefore, the p_n and q_n of (112, 114) are a solution to (116-122). Since the process forms an aperiodic, irreducible, ergodic Markov chain (imbedded at service completion epochs), this solution is unique.

The Case $c > x > 1$. In the steady state with $c > x > 1$, Equations (82-88) become

$$0 = -N\lambda p_o + \mu p_1; \tag{123}$$

$$0 = -(N\lambda+n\mu)p_n + N\lambda p_{n-1} + (n+1)\mu p_{n+1}, \qquad n=1,2,3,\cdots,x-1; \tag{124}$$

$$0 = -(N\lambda+x\mu)p_x + N\lambda p_{x-1} + (x+1)\mu p_{x+1} + (x+1)\mu q_{x+1}; \tag{125}$$

$$0 = -(N\lambda+n\mu)p_n + N\lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad n=x+1,x+2,x+3,\cdots,c-1; \quad (126)$$

$$0 = -(N\lambda+c\mu)p_n + N\lambda p_{n-1} + c\mu p_{n+1}, \quad n=c,c+1,c+2,\cdots,S-1; \quad (127)$$

$$0 = -(N\lambda+c\mu)p_S + N\lambda p_{S-1}; \quad (128)$$

$$0 = -n\mu q_n + (n+1)\mu q_{n+1}, \quad n=x+1,x+2,x+3,\cdots,c-1; \quad (129)$$

$$0 = -c\mu q_n + c\mu q_{n+1}, \quad n=c,c+1,c+2,\cdots,S; \quad (130)$$

$$0 = -c\mu q_{S+1} + N\lambda p_S. \quad (131)$$

It will be found that the solution is much more difficult to obtain for the case $x < c$. The underlying causes of difficulty are that servicing does not proceed at a uniform rate throughout an extended service period and that the system is restarted while the service center is operating at less than full capacity. Mathematically, this difficulty can be expressed as the necessity of finding a general solution to Equations (126).

Let us first collect the simple results. Equations (123-124) may be evaluated recursively to obtain

$$p_n = \frac{1}{n!}\left(\frac{N\lambda}{\mu}\right)^n p_o, \quad n=1,2,3,\cdots,x. \quad (132)$$

Further, using the argument centered around Equations (101-107), we can immediately write for Equations (127-128),

$$p_n = A_3\left[\left(\frac{N\lambda}{c\mu}\right)^n - \left(\frac{N\lambda}{c\mu}\right)^{S+1}\right], \quad n=c,c+1,c+2,\cdots,S; \tag{133}$$

where A_3 is an unknown constant.

It is obvious from Equations (130-131) that

$$q_n = q_{S+1} = \left(\frac{N\lambda}{c\mu}\right)p_S, \quad n=c,c+1,c+2,\cdots,S+1. \tag{134}$$

Further, solving Equations (129) recursively, it follows that

$$q_n = \left(\frac{c}{n}\right)q_c = \left(\frac{c}{n}\right)q_{S+1} = \left(\frac{N\lambda}{n\mu}\right)p_S, \tag{135}$$

$$n=x+1,x+2,x+3,\cdots,c.$$

In particular,

$$q_{x+1} = \frac{N\lambda}{(x+1)\mu}\,p_S \tag{136}$$

so that

$$(x+1)\mu q_{x+1} = N\lambda A_3\left(\frac{N\lambda}{c\mu}\right)^S\left(1 - \frac{N\lambda}{c\mu}\right) \tag{137}$$

in view of (133).

It remains to treat Equations (125-126). It is possible to approach Equation (126) recursively starting with its upper bound and using (133). It may also be approached from the lower bound if first

(132) and (137) are used in (125) to determine an expression for p_{x+1}. However, in this instance, the recursive approach rapidly gets mired in computations of increasing complexity before it is possible to ascertain a pattern. What is needed is a *general* solution for the difference equation,

$$Hy_k \equiv (k+1)y_{k+1} - (\alpha+k)y_k + \alpha y_{k-1} = 0, \tag{138}$$

where H is an operator defined by the equation and α is a constant. (Equation (138) is equivalent to (126) with $(N\lambda/\mu) = \alpha$.)

Equation (138) has occurred previously in the discussion. We know that $\alpha^k/k!$ is a solution for a particular set of boundary conditions. (See Equations (123-124,132).) Let us apply this knowledge and attempt to find functions F_k such that

$$y_k = F_k \alpha^k/k! \tag{139}$$

is another solution to (138). We suppose that F_k is of the form

$$F_k = \sum_{j=k}^{U} f_j, \tag{140}$$

where U is some upper bound on k. Substituting (140) into (138), we find

$$0 = (k+1)\frac{\alpha^{k+1}}{(k+1)!}\sum_{j=k+1}^{U} f_j - (\alpha+k)\frac{\alpha^k}{k!}\sum_{j=k}^{U} f_j + \alpha\frac{\alpha^{k-1}}{(k-1)!}\sum_{j=k-1}^{U} f_j$$

$$= (k+1)\,\frac{\alpha^{k+1}}{(k+1)!}\sum_{j=k+1}^{U} f_j - (\alpha+k)\,\frac{\alpha^k}{k!}\left[f_k + \sum_{j=k+1}^{U} f_j\right]$$

$$+ \alpha\,\frac{\alpha^{k-1}}{(k-1)!}\left[f_{k-1} + f_k + \sum_{j=k+1}^{U} f_j\right]. \tag{141}$$

A rearrangement of terms yields the form

$$0 = -(\alpha+k)\,\frac{\alpha^k}{k!}\,f_k + \frac{\alpha^k}{(k-1)!}\,(f_{k-1}+f_k) + \left[\sum_{j=k+1}^{U} f_j\right] H\left(\frac{\alpha^k}{k!}\right)$$

$$= (\alpha^k/k!)[-\alpha f_k + k f_{k-1}]. \tag{142}$$

Therefore,

$$f_k = (k/\alpha) f_{k-1}, \tag{143}$$

from which it follows that

$$F_k = \sum_{j=k}^{U} k!\,\alpha^{-k} f_o. \tag{144}$$

Accordingly, we can write the general solution to (138) in the form

$$y_k = \left(\frac{\alpha^k}{k!}\right)\left[B_1 + B_2 \sum_{j=k}^{U} k!\,\alpha^{-k}\right], \tag{145}$$

where B_1 and B_2 are unknown constants and U is an upper limit to be specified. Note that any change in the value of U can be accommodated by a change in the value of B_1.

Applying this result to Equation (126), we obtain

$$p_n = \frac{1}{n!}\left(\frac{N\lambda}{\mu}\right)^n \left[A_4 + A_5 \sum_{j=n}^{c} j! \left(\frac{\mu}{N\lambda}\right)^j\right], \tag{146}$$

$$n=x,x+1,x+2,\cdots,c.$$

We require (146) to have continuity with (132) and (133). Thus, we require

$$\frac{1}{x!}\left(\frac{N\lambda}{\mu}\right)^x p_o = \frac{1}{x!}\left(\frac{N\lambda}{\mu}\right)^x \left[A_4 + A_5 \sum_{j=x}^{c} j! \left(\frac{\mu}{N\lambda}\right)^j\right], \tag{147}$$

$$A_3\left[\left(\frac{N\lambda}{c\mu}\right)^c - \left(\frac{N\lambda}{c\mu}\right)^{S+1}\right] = \frac{1}{c!}\left(\frac{N\lambda}{\mu}\right)^c\left[A_4 + A_5 c!\left(\frac{\mu}{N\lambda}\right)^c\right]. \tag{148}$$

The conditions (147-148) allow us to express A_4 and A_5 in terms of the constants p_o and A_3. These values are

$$A_4 = \left\{-c!\left(\frac{\mu}{N\lambda}\right)^c p_o + c!c^{-c}A_3\left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-c+1}\right]\left[\sum_{j=x}^{c} j!\left(\frac{\mu}{N\lambda}\right)^j\right]\right\}\left[\sum_{j=x}^{c-1} j!\left(\frac{\mu}{N\lambda}\right)^j\right]^{-1}, \tag{149}$$

$$A_5 = \left\{p_o - c!c^{-c}A_3\left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-c+1}\right]\right\}\left[\sum_{j=x}^{c-1} j!\left(\frac{\mu}{N\lambda}\right)^j\right]^{-1}. \tag{150}$$

Now, using (132) and (137) in (125), we find

$$0 = -(N\lambda+x\mu)\frac{1}{x!}\left(\frac{N\lambda}{\mu}\right)^{x} p_o + N\lambda\frac{1}{(x-1)!}\left(\frac{N\lambda}{\mu}\right)^{x} p_o + (x+1)\mu p_{x+1}$$

$$+ N\lambda A_3\left(\frac{N\lambda}{c\mu}\right)^{S}\left(1 - \frac{N\lambda}{c\mu}\right)$$

$$= -N\lambda\left(\frac{N\lambda}{\mu}\right)^{x}\frac{p_o}{x!} + N\lambda A_3\left(\frac{N\lambda}{c\mu}\right)^{S}\left(1 - \frac{N\lambda}{c\mu}\right) + (x+1)\mu p_{x+1}. \tag{151}$$

From (146), we have

$$(x+1)\mu p_{x+1} = \frac{N\lambda}{x!}\left(\frac{N\lambda}{\mu}\right)^{x}\left[A_4 + A_5\sum_{j=x+1}^{c} j!\left(\frac{\mu}{N\lambda}\right)^{j}\right]$$

$$= \frac{N\lambda}{x!}\left(\frac{N\lambda}{\mu}\right)^{x}\left[p_o - A_5(x!)\left(\frac{\mu}{N\lambda}\right)^{x}\right]$$

$$= \frac{N\lambda}{x!}\left(\frac{N\lambda}{\mu}\right)^{x} p_o - N\lambda A_5 \tag{152}$$

in view of (147). Substituting (150) into (152) and the result into (151), we obtain

$$0 = -N\lambda\left(\frac{N\lambda}{\mu}\right)^{x}\frac{p_o}{x!} + N\lambda A_3\left(\frac{N\lambda}{c\mu}\right)^{S}\left(1 - \frac{N\lambda}{c\mu}\right) + \frac{N\lambda}{x!}\left(\frac{N\lambda}{\mu}\right)^{x} p_o$$

$$-N\lambda\left\{p_o - c!c^{-c}A_3\left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-c+1}\right]\right\}\left[\sum_{j=x}^{c-1} j!\left(\frac{\mu}{N\lambda}\right)^{j}\right]^{-1}, \tag{153}$$

or

$$0 = A_3\left(\frac{N\lambda}{c\mu}\right)^{S}\left(1 - \frac{N\lambda}{c\mu}\right)\left[\sum_{j=x}^{c-1} j!\left(\frac{\mu}{N\lambda}\right)^{j}\right] - p_o$$

$$+ c!c^{-c}A_3\left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-c+1}\right]. \qquad (154)$$

Hence,

$$A_3 = p_o\left\{c!c^{-c}\left[1 - \left(\frac{N\lambda}{c\mu}\right)^{S-c+1}\right] + \left(\frac{N\lambda}{c\mu}\right)^S\left(1 - \frac{N\lambda}{c\mu}\right)\sum_{j=x}^{c-1} j!\left(\frac{\mu}{N\lambda}\right)^j\right\}^{-1}. \qquad (155)$$

We now define p_{S+1} as the total probability that the system is in the failed state. Thus,

$$p_{S+1} = \sum_{n=x+1}^{S+1} q_n = \left(\frac{N\lambda}{\mu}\right)p_S\left[\sum_{n=x+1}^{c}\frac{1}{n} + \sum_{n=c+1}^{S+1}\frac{1}{c}\right]$$

$$= \left(\frac{N\lambda}{\mu}\right)p_S\left[\frac{S-c+1}{c} + \sum_{n=x+1}^{c}\frac{1}{n}\right]. \qquad (156)$$

The complete solution is given by Equations (132-133,146,156). Grouping these results, we have for $c > x > 1$,

$$\left.\begin{aligned}
p_n &= \frac{1}{n!}\left(\frac{N\lambda}{\mu}\right)^n p_o, && n=1,2,3,\cdots,x;\\
p_n &= \frac{1}{n!}\left(\frac{N\lambda}{\mu}\right)^n\left[A_4 + A_5\sum_{j=n}^{c} j!\left(\frac{\mu}{N\lambda}\right)^j\right], && n=x,x+1,x+2,\cdots,c;\\
p_n &= A_3\left[\left(\frac{N\lambda}{c\mu}\right)^n - \left(\frac{N\lambda}{c\mu}\right)^{S+1}\right], && n=c,c+1,c+2,\cdots,S;\\
p_{S+1} &= A_3\left(\frac{N\lambda}{c\mu}\right)^{S+1}\left[1 - \left(\frac{N\lambda}{c\mu}\right)\right]\left[(S-c+1) + \sum_{n=x+1}^{c}\frac{c}{n}\right];
\end{aligned}\right\} \qquad (157)$$

where A_3, A_4, and A_5 are constants defined in terms of p_o by Equations (149-150,155) and where p_o is to be evaluated using the normalizing condition (115).

The Case $x = 1$. In the steady state with $x = 1$, Equations (82-88) become

$$0 = -N\lambda p_o + \mu p_1; \tag{158}$$

$$0 = -(N\lambda+\mu)p_1 + N\lambda p_o + 2\mu p_2 + 2\mu q_2; \tag{159}$$

$$0 = -(N\lambda+n\mu)p_n + N\lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad n=2,3,4,\cdots,c-1; \tag{160}$$

$$0 = -(N\lambda+c\mu)p_n + N\lambda p_{n-1} + c\mu p_{n+1}, \quad n=c,c+1,c+2,\cdots,S-1; \tag{161}$$

$$0 = -(N\lambda+c\mu)p_S + N\lambda p_{S-1}; \tag{162}$$

$$0 = -n\mu q_n + (n+1)\mu q_{n+1}, \quad n=2,3,4,\cdots,c-1; \tag{163}$$

$$0 = -c\mu q_n + c\mu q_{n+1}, \quad n=c,c+1,c+2,\cdots,S; \tag{164}$$

$$0 = -c\mu q_{S+1} + N\lambda p_S. \tag{165}$$

It can be verified by substitution that Equations (158-165) are satisfied by the p_n and q_n of (134-135,157). Therefore, the p_n and q_n of (134-135,157) are a solution to (158-165). Since the process is an aperiodic, irreducible, ergodic Markov chain, this solution is unique.

The Case $x = 0$. When $x = 0$, the operating procedure is to make all components of the system fully functional during an extended service period. The steady-state equations are

$$0 = -N\lambda p_o + \mu p_1 + \mu q_1; \tag{166}$$

$$0 = -(N\lambda+n\mu)p_n + N\lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad n=1,2,3,\cdots,c-1; \tag{167}$$

$$0 = -(N\lambda+c\mu)p_n + N\lambda p_{n-1} + c\mu p_{n+1}, \quad n=c,c+1,c+2,\cdots,S-1; \tag{168}$$

$$0 = -(N\lambda+c\mu)p_S + N\lambda p_{S-1}; \tag{169}$$

$$0 = -n\mu q_n + (n+1)\mu q_{n+1}, \qquad n=1,2,3,\cdots,c-1; \tag{170}$$

$$0 = -c\mu q_n + c\mu q_{n+1}, \qquad n=c,c+1,c+2,\cdots,S; \tag{171}$$

$$0 = -c\mu q_{S+1} + N\lambda p_S. \tag{172}$$

Again it can be verified that the p_n and q_n of (134-135,157) give the unique solution.

A Time-Dependent Model

Introduction

An expression for the Laplace transforms $P_n^e(s)$ of the time-dependent state probabilities $P_n(t)$ for the single-server version of Taylor and Jackson's model [232] will now be derived. It is assumed that the system is initially in the state where all units are operative. It will be shown that the $P_n^e(s)$ are of the form $r(s)/q(s)$, where $r(s)$ and $q(s)$ are real polynomials, and thus are amenable to partial fraction expansion and subsequent term-by-term inversion in a straightforward manner. A discussion of the prospects of finding time-dependent solutions to other repairman problems with spares will follow.

Taylor and Jackson's model is a special case of the general birth-and-death model described in Chapter I. It will be recalled that, for this general model, a sufficient condition for the existence of a unique, time-dependent solution satisfying the regularity condition (8) is that the sequences of coefficients $\{\lambda_n\}$, $\{\mu_n\}$ be bounded. We will assume that λ and μ in Equations (40) are finite positive. Since there is a finite number of states, boundedness of $\{\lambda_n\}$, $\{\mu_n\}$ is assured and there must be a unique, regular solution.

Equations of the Model

For the single-server case with W spares, the time-dependent equations of Taylor and Jackson's model become

$$\left.\begin{aligned}
\frac{dP_o(t)}{dt} &= -N\lambda P_o(t) + \mu P_1(t);\\
\frac{dP_n(t)}{dt} &= -(N\lambda+\mu)P_n(t) + N\lambda P_{n-1}(t) + \mu P_{n+1}(t),\\
&\qquad n=1,2,3,\cdots,W;
\end{aligned}\right\} \quad (173)$$

$$\left.\frac{dP_{W+1}(t)}{dt} = -\mu P_{W+1}(t) + N\lambda P_W(t).\right\}$$

Using the Kronecker delta, Equation (90), the initial conditions can be expressed as

$$P_n(0) = \delta_{n,k}, \quad n=0,1,2,\cdots,W+1; \tag{174}$$

if there are k failed machines in the system at time 0. With this notation, the unilateral Laplace transforms of Equations (173) can be written as

$$\left.\begin{aligned} -\delta_{o,k} &= -(s+N\lambda)P_o^e(s) + \mu P_1^e(s); \\ -\delta_{n,k} &= -(s+N\lambda+\mu)P_n^e(s) + N\lambda P_{n-1}^e(s) + \mu P_{n+1}^e(s), \\ &\qquad\qquad n=1,2,3,\cdots,W; \\ -\delta_{W+1,k} &= -(s+\mu)P_{W+1}^e(s) + N\lambda P_W^e(s). \end{aligned}\right\} \tag{175}$$

<u>Solution for the Case k = 0</u>

An obvious solution rationale is suggested by the form of Equations (175). For given s, the system (175) is a set of second-order, linear, nonhomogeneous difference equations in n with constant coefficients. General procedures exist for solving such equations (see, e.g., [83,140]), so it should be a straightforward matter to determine the $P_n^e(s)$ in (175) for an arbitrary specification of k. We will demonstrate the procedure for the case $k = 0$.

When $k = 0$, Equations (175) become

$$-1 = -(s+N\lambda)P_o^e(s) + \mu P_1^e(s); \tag{176}$$

$$0 = -(s+N\lambda+\mu)P_n^e(s) + N\lambda P_{n-1}^e(s) + \mu P_{n+1}^e(s), \tag{177}$$

$$n=1,2,3,\cdots,W;$$

$$0 = -(s+\mu)P_{W+1}^e(s) + N\lambda P_W^e(s). \tag{178}$$

For given s, Equation (177) is a second-order, linear, homogeneous difference equation in n of the form

$$0 = a_1 x_{n+1} + a_o x_n + a_{-1} x_{n-1}, \tag{179}$$

where the coefficients a_1, a_o, a_{-1} are known constants. Equations (179) admit to solutions of the form $C\beta^n$, where C is an arbitrary constant to be determined from boundary conditions. Thus, in (177), we make the substitution

$$P_n^e(s) = C(s)\beta(s)^n \tag{180}$$

and eliminate common factors from both sides of the result to obtain the condition

$$0 = -(s+N\lambda+\mu)\beta(s) + N\lambda + \mu\beta(s)^2. \tag{181}$$

Solving (181), we find two roots,

$$\beta_1(s) = [1/(2\mu)]\ \{(s+N\lambda+\mu) + [(s+N\lambda+\mu)^2 - 4N\lambda\mu]^{1/2}\}, \qquad (182)$$

$$\beta_2(s) = [1/(2\mu)]\ \{(s+N\lambda+\mu) - [(s+N\lambda+\mu)^2 - 4N\lambda\mu]^{1/2}\}, \qquad (183)$$

Therefore, letting $C_1(s)$ and $C_2(s)$ denote arbitrary constants, we have a general solution in the form

$$P_n^e(s) = C_1(s)\beta_1(s)^n + C_2(s)\beta_2(s)^n, \quad n=0,1,2,\cdots,W+1; \qquad (184)$$

although it is recognizable that, due to the complementary relationship of $\beta_1(s)$ and $\beta_2(s)$, alternate representations (e.g., involving hyperbolic cosines and sines) are possible. To obtain the values of $C_1(s)$ and $C_2(s)$, we substitute (184) into the boundary Equations (176,178) to find

$$1 = C_1(s)[s+N\lambda-\mu\beta_1(s)] + C_2(s)[s+N\lambda-\mu\beta_2(s)], \qquad (185)$$

$$0 = C_1(s)\beta_1(s)^W[N\lambda-(s+\mu)\beta_1(s)] + C_2(s)\beta_2(s)^W[N\lambda-(s+\mu)\beta_2(s)]. \qquad (186)$$

The values are

$$C_1(s) = \beta_2(s)^W[N\lambda-(s+\mu)\beta_2(s)]\ \{\beta_2(s)^W[s+N\lambda-\mu\beta_1(s)][N\lambda-(s+\mu)\beta_2(s)]$$

$$- \beta_1(s)^W[s+N\lambda-\mu\beta_2(s)][N\lambda-(s+\mu)\beta_1(s)]\}^{-1}, \qquad (187)$$

$$C_2(s) = -\beta_1(s)^W[N\lambda-(s+\mu)\beta_1(s)]\ \{\beta_2(s)^W[s+N\lambda-\mu\beta_1(s)][N\lambda-(s+\mu)\beta_2(s)]$$

$$-\ \beta_1(s)^W[s+N\lambda-\mu\beta_2(s)][N\lambda-(s+\mu)\beta_1(s)]\}^{-1}. \qquad (188)$$

However, noting that two possible rearrangements of (181) are

$$[s+N\lambda-\mu\beta(s)] = [N\lambda-\mu\beta(s)]/\beta(s), \qquad (189)$$

$$N\lambda - (s+\mu)\beta(s) = \beta(s)[N\lambda-\mu\beta(s)], \qquad (190)$$

we can write these expressions in the forms

$$C_1(s) = \beta_2(s)^{W+1}[N\lambda-\mu\beta_2(s)]/A(s), \qquad (191)$$

$$C_2(s) = -\beta_1(s)^{W+1}[N\lambda-\mu\beta_1(s)]/A(s), \qquad (192)$$

where

$$A(s) = [N\lambda-\mu\beta_1(s)][N\lambda-\mu\beta_2(s)][\beta_2(s)^{W+2}-\beta_1(s)^{W+2}][\beta_1(s)\beta_2(s)]^{-1}. \qquad (193)$$

But from (182-183), we have

$$\beta_1(s)\beta_2(s) = N\lambda/\mu, \qquad (194)$$

$$\beta_1(s) + \beta_2(s) = (s+N\lambda+\mu)/\mu. \qquad (195)$$

So A(s) can be reduced to

$$A(s) = \{(N\lambda)^2 - N\lambda\mu[\beta_1(s)+\beta_2(s)] + \mu^2\beta_1(s)\beta_2(s)\}$$

$$\cdot\, [\beta_2(s)^{W+2} - \beta_1(s)^{W+2}][\beta_1(s)\beta_2(s)]^{-1}$$

$$= [(N\lambda)^2 - N\lambda(s+N\lambda+\mu) + \mu N\lambda][\beta_2(s)^{W+2} - \beta_1(s)^{W+2}][N\lambda/\mu]^{-1}$$

$$= -\mu s[\beta_2(s)^{W+2} - \beta_1(s)^{W+2}]. \tag{196}$$

Hence,

$$C_1(s) = \frac{\beta_2(s)^{W+1}[N\lambda-\mu\beta_2(s)]}{\mu s[\beta_1(s)^{W+2}-\beta_2(s)^{W+2}]}, \tag{197}$$

$$C_2(s) = \frac{-\beta_1(s)^{W+1}[N\lambda-\mu\beta_1(s)]}{\mu s[\beta_1(s)^{W+2}-\beta_2(s)^{W+2}]}. \tag{198}$$

Substituting (197-198) into (184) and again using (194) yields, for $k = 0$,

$$P_n^e(s) = \left(\frac{N\lambda}{\mu}\right)^n \left[\frac{\mu[\beta_1(s)^{W-n+2}-\beta_2(s)^{W-n+2}] - N\lambda[\beta_1(s)^{W-n+1}-\beta_2(s)^{W-n+1}]}{\mu s[\beta_1(s)^{W+2}-\beta_2(s)^{W+2}]}\right] \tag{199}$$

$$n=0,1,2,\cdots,W+1;$$

where $\beta_1(s)$ and $\beta_2(s)$ are given by Equations (182-183).

It now remains to invert the $P_n^e(s)$ to find the $P_n(t)$. We will indicate how this may be done. Let $[\![x]\!]$ denote the largest integer that does not exceed the real number x,* and let

$$a = (s+N\lambda+\mu)/(2\mu), \tag{200}$$

$$b = [(s+N\lambda+\mu)^2-4N\lambda\mu]^{1/2}/(2\mu). \tag{201}$$

Then, for any positive integer m,

$$\begin{aligned}\beta_1(s)^m - \beta_2(s)^m &= (a+b)^m - (a-b)^m \\ &= \sum_{j=0}^{m} \binom{m}{j} a^{m-j}[b^j-(-b)^j] \\ &= 2b \sum_{i=0}^{I} \binom{m}{2i+1} a^{m-2i-1}b^{2i} \\ &= 2b \sum_{i=0}^{I} \binom{m}{2i+1} a^{m-2i-1}\,[a^2-(N\lambda/\mu)]^i, \end{aligned} \tag{202}$$

where $I = [\![(m-1)/2]\!]$. It can be seen that $[\beta_1(s)^m-\beta_2(s)^m]/b$ is a real polynomial in a, and hence in s, of degree m-1. Further, this polynomial is the product of a^{m-2I-1} (i.e., 1 if m is odd and a if m is even) and a real polynomial in s^2 of degree I. It follows that Equation (199) has the form

$$P_n^e(s) = \frac{R(s;W-n+1)}{sR(s;W+1)} = \frac{R(s;W-n+1)}{sR(s^2;J)a^{W-2J+1}}, \tag{203}$$

*For example, $[\![3.72]\!] = 3$ and $[\![2.0]\!] = 2$.

where $J = [\![(W+1)/2]\!]$ and where $R(y;k)$ denotes a real polynomial in y of degree k. Since its denominator has J pairs of conjugate roots, $P_n^e(s)$ has a partial fraction expansion of at most J + 2 terms, each of which can be inverted in a straightforward manner using standard tables of inverse Laplace transforms. (See [140], pp.21-22,230.)

Solution for Other Cases

Much of the solution for cases with $k \geq 1$ can be borrowed from the previous case. If $k = W + 1$, Equations (175) take the form

$$0 = -(s+N\lambda)P_o^e(s) + \mu P_1^e(s); \tag{204}$$

$$0 = -(s+N\lambda+\mu)P_n^e(s) + N\lambda P_{n-1}^e(s) + \mu P_{n+1}^e(s), \tag{205}$$

$$n=1,2,3,\cdots,W;$$

$$-1 = -(s+\mu)P_{W+1}^e(s) + N\lambda P_W^e(s). \tag{206}$$

The general solution to Equation (205) is given by (184) with $\beta_1(s)$ and $\beta_2(s)$ specified by (182-183). The unknowns $C_1(s)$ and $C_2(s)$ are to be determined using the boundary conditions (204,206).

For $1 \leq k \leq W$, Equations (175) become

$$0 = -(s+N\lambda)P_o^e(s) + \mu P_1^e(s); \tag{207}$$

$$0 = -(s+N\lambda+\mu)P_n^e(s) + N\lambda P_{n-1}^e(s) + \mu P_{n+1}^e(s), \tag{208}$$

$$n=1,2,3,\cdots,k-1;$$

$$-1 = -(s+N\lambda+\mu)P_k^e(s) + N\lambda P_{k-1}^e(s) + \mu P_{k+1}^e(s); \tag{209}$$

$$0 = -(s+N\lambda+\mu)P_n^e(s) + N\lambda P_{n-1}^e(s) + \mu P_{n+1}^e(s), \tag{210}$$

$$n=k+1,k+2,k+3,\cdots,W;$$

$$0 = -(s+\mu)P_{W+1}^e(s) + N\lambda P_W^e(s); \tag{211}$$

where Equations (208) are omitted if $k = 1$ and Equations (210) are omitted if $k = w$. The general solution to Equations (208,210) is

$$P_n^e(s) = C_3(s)\beta_1(s)^n + C_4(s)\beta_2(s)^n, \quad n=0,1,2,\cdots,k; \tag{212}$$

$$P_n^e(s) = C_5(s)\beta_1(s)^n + C_6(s)\beta_2(s)^n, \quad n=k,k+1,k+2,\cdots,W+1; \tag{213}$$

where $\beta_1(s)$ and $\beta_2(s)$ are specified in (182-183) and where $C_3(s)$, $C_4(s)$, $C_5(s)$, and $C_6(s)$ are unknown functions of s and independent of n. To determine $C_3(s)$, $C_4(s)$, $C_5(s)$, and $C_6(s)$, we would use the boundary conditions (207,209,211) and the requirement of continuity between (212,213) at $n = k$. The relation,

$$\frac{1}{s} = \sum_{n=0}^{W+1} P_n^e(s), \tag{214}$$

which is the transform of

$$1 = \sum_{n=0}^{W+1} P_n(t), \tag{215}$$

is also satisfied.

Extension to Other Problems

The prospects for extension of the method to other repairman problems with spares are not generally encouraging. It is seen that the method requires the successful execution of three principal steps: (1) obtaining the Laplace transform of the differential-difference equations defining the system, (2) solving the resultant difference equations to obtain the $P_n^e(s)$, and (3) manipulating the expressions for the $P_n^e(s)$ into a form amenable to inverse transformation. In almost all cases--certainly for all of the models discussed in this dissertation--the first step can be easily accomplished. However, the second and third steps are entirely a different matter.

The second step requires the solution of a set of difference equations. In the simple problem just treated, these equations (175) had the desirable property of possessing coefficients independent of the differencing variable so that a standard procedure (see Equations (179-183) could be used to find a general solution. This case was atypical: Nearly all repairman problems with spares give rise to transformed difference equations with variable coefficients.* Unfortunately, there are no standard procedures for determining general solutions to difference equations with variable coefficients** so that successful

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*In fact, of all the repairman problems with spares discussed in this dissertation, only the problem treated here, Taylor and Jackson's simple single-server system, and one or two others have this constant-coefficient property.

**There are standard methods for constructing *particular* solutions to differential equations with variable coefficients (e.g., see completion of the second step will be a nontrivial matter in even slightly more complex problems. The third step requires little comment. A more complex problem will give rise to more complex expressions for the $P_n^e(s)$ that are accordingly less amenable to manipulation into a form suitable to inverse transformation.

Therefore, it appears that problems of execution, primarily in the second and third steps, will prevent a ready extension of the method to other repairman problems with spares. The prognosis is also pessimistic from the standpoint of using other existing methods. In the 60 years of growth of queueing theory, there has been a relative dirth of papers on transient models. Further, there have been no papers at all on transient solutions to regular repairman problems during their 35-year history.

A Model Involving Ancillary Duties

An Operator with Ancillary Duties

The situation of interest is that where a single operator has charge of a group of automatic machines in the sense of the classical machine-tending models discussed in Chapter II. The stoppage and clearing times on a machine are assumed to be negative exponential variates with mean values $1/\lambda$ and $1/\mu$, respectively. The operator is occasionally called upon to perform ancillary duties and at such times must leave immediately to attend to these duties irrespective of the state of the machines. The intervals separating these calls are assumed to follow

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Appendix). All of these presume that the homogeneous solution has already been found.

a negative-exponential distribution and have average length $1/\nu$. The duration of a period of ancillary work is likewise assumed to be a negative exponential variate, with mean duration $1/\eta$.

The machines operate as a battery: It requires N functional machines to keep the system operative. However, there are a total of N + S machines in the system and, when a machine stops, a standby spare (if one is available) is automatically activated without the necessity of intervention by the operator. In summary, the process is that studied by Taylor and Jackson [232], but includes the additional feature of ancillary duties for the single operator.

<u>Model Formulation</u>

Let $P_n(t)$ and $Q_n(t)$ be the probabilities that at time t there are n failed machines in the system with the operator present and absent, respectively. The unconditional probability of there being n failed machines at time t is $P_n(t) + Q_n(t)$. Using the usual approach of considering the possible happenings in a small interval of time Δt (see Chapter I), we obtain the relations

$$P_o(t+\Delta t) = (1-\nu\Delta t)\{(1-N\lambda\Delta t)P_o(t) + (1-N\lambda\Delta t)(\mu\Delta t)P_1(t)\}$$

$$+ (\eta\Delta t)(1-N\lambda\Delta t)Q_o(t) + O(\Delta t); \qquad (216)$$

$$P_n(t+\Delta t) = (1-\nu\Delta t)\{(1-N\lambda\Delta t)(1-\mu\Delta t)P_n(t)$$

$$+ (N\lambda\Delta t)(1-\mu\Delta t)P_{n-1}(t) + (1-N\lambda\Delta t)(\mu\Delta t)P_{n+1}(t)\}$$

$$+ (\eta\Delta t)(1-N\lambda\Delta t)Q_n(t) + O(\Delta t), \quad n=1,2,3,\cdots,S; \tag{217}$$

$$P_{S+1}(t+\Delta t) = (1-\nu\Delta t)\{(1-\mu\Delta t)P_{S+1}(t) + (N\lambda\Delta t)(1-\mu\Delta t)P_S(t)\}$$

$$+ (\eta\Delta t)Q_{S+1}(t) + O(\Delta t); \tag{218}$$

$$Q_o(t+\Delta t) = (1-\eta\Delta t)(1-N\lambda\Delta t)Q_o(t) + (\nu\Delta t)(1-N\lambda\Delta t)P_o(t) + O(\Delta t); \tag{219}$$

$$Q_n(t+\Delta t) = (1-\eta\Delta t)\{(1-N\lambda\Delta t)Q_n(t) + (N\lambda\Delta t)Q_{n-1}(t)\}$$

$$+ (\nu\Delta t)(1-N\lambda\Delta t)(1-\mu\Delta t)P_n(t) + O(\Delta t), \quad n=1,2,3,\cdots,S; \tag{220}$$

$$Q_{S+1}(t+\Delta t) = (1-\eta\Delta t)\{Q_{S+1}(t) + (N\lambda\Delta t)Q_S(t)\}$$

$$+ (\nu\Delta t)(1-\mu\Delta t)P_{S+1}(t) + O(\Delta t); \tag{221}$$

where $O(\Delta t)$ denotes the inclusion of quantities of smaller order of magnitude than Δt.

Taking the limit as $\Delta t \to 0$ yields the dynamic equations of the system,

$$\frac{dP_o(t)}{dt} = -(N\lambda+\nu)P_o(t) + \mu P_1(t) + \eta Q_o(t); \tag{222}$$

$$\frac{dP_n(t)}{dt} = -(N\lambda+\mu+\nu)P_n(t) + N\lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

$$+ \eta Q_n(t), \quad n=1,2,3,\cdots,S; \tag{223}$$

$$\frac{dP_{S+1}(t)}{dt} = -(\mu+\nu)P_{S+1}(t) + N\lambda P_S(t) + \eta Q_{S+1}(t); \tag{224}$$

$$\frac{dQ_o(t)}{dt} = -(N\lambda+\eta)Q_o(t) + \nu P_o(t); \tag{225}$$

$$\frac{dQ_n(t)}{dt} = -(N\lambda+\eta)Q_n(t) + N\lambda Q_{n-1}(t) + \nu P_n(t), \quad n=1,2,3,\cdots,S; \tag{226}$$

$$\frac{dQ_{S+1}(t)}{dt} = -\eta Q_{S+1}(t) + N\lambda Q_S(t) + \nu P_{S+1}(t). \tag{227}$$

Let p_n and q_n be the stationary probabilities associated with $P_n(t)$ and $Q_n(t)$, respectively. Then, the steady-state equations are

$$0 = -(N\lambda+\nu)p_o + \mu p_1 + \eta q_o; \tag{228}$$

$$0 = -(N\lambda+\mu+\nu)p_n + N\lambda p_{n-1} + \mu p_{n+1} + \eta q_n, \quad n=1,2,3,\cdots,S; \tag{229}$$

$$0 = -(\mu+\nu)p_{S+1} + N\lambda p_S + \eta q_{S+1}; \tag{230}$$

$$0 = -(N\lambda+\eta)q_o + \nu p_o; \tag{231}$$

$$0 = -(N\lambda+\eta)q_n + N\lambda q_{n-1} + \nu p_n, \quad n=1,2,3,\cdots,S; \tag{232}$$

$$0 = -\eta q_{S+1} + N\lambda q_S + \nu p_{S+1}. \tag{233}$$

The p_n and q_n must satisfy the certainty condition for a regular solution,

$$\sum_{n=0}^{S+1} p_n + \sum_{n=0}^{S+1} q_n = 1. \tag{234}$$

Steady-State Solution

From (229), we obtain

$$q_n = \eta^{-1}[-\mu p_{n+1} + (N\lambda+\mu+\nu)p_n - N\lambda p_{n-1}], \tag{235}$$

$$n=1,2,3,\cdots,S;$$

and

$$q_{n-1} = \eta^{-1}[-\mu p_n + (N\lambda+\mu+\nu)p_{n-1} - N\lambda p_{n-2}], \tag{236}$$

$$n=2,3,4,\cdots,S+1.$$

Substituting (235-236) into (232) yields

$$0 = -(N\lambda+\eta)\eta^{-1}[-\mu p_{n+1} + (N\lambda+\mu+\nu)p_n - N\lambda p_{n-1}]$$

$$+ N\lambda\eta^{-1}[-\mu p_n + (N\lambda+\mu+\nu)p_{n-1} - N\lambda p_{n-2}] + \nu p_n, \tag{237}$$

$$n=2,3,4,\cdots,S.$$

Multiplying this through by η and rearranging terms results in

$$0 = \mu(N\lambda+\eta)p_{n+1} - [N\lambda(N\lambda+2\mu+\eta+\nu) + \eta\nu]p_n$$

$$+ N\lambda(2N\lambda+\mu+\eta+\nu)p_{n-1} - (N\lambda)^2 p_{n-2}, \qquad (238)$$

$$n=2,3,4,\cdots,S.$$

Equation (238) can be recognized as being a third-order, homogeneous, linear difference equation in n with constant coefficients. Accordingly, it admits to solutions of the form

$$p_n = C\beta^n, \qquad (239)$$

where C is an arbitrary constant. Substituting (239) into (238) and dividing through by common factors, we obtain a cubic equation in β,

$$0 = \mu(N\lambda+\eta)\beta^3 - [N\lambda(N\lambda+2\mu+\eta+\nu) + \eta\nu]\beta^2$$

$$+ N\lambda(2N\lambda+\mu+\eta+\nu)\beta - (N\lambda)^2 \qquad (240)$$

Let the roots of (240) be denoted by β_1, β_2, and β_3. Direct substitution will show that $\beta = 1$ is a solution of (240), so write

$$\beta_1 = 1, \qquad (241)$$

and divide (240) through by $(\beta-1)$ to obtain the quadratic which β_2 and β_3 must satisfy, namely,

$$0 = \mu(N\lambda+\eta)\beta^2 - N\lambda(N\lambda+\mu+\nu+\eta)\beta + (N\lambda)^2. \tag{242}$$

Solving, we find

$$\beta_2 = \frac{N\lambda}{2\mu A}(B + \sqrt{B^2-4\mu A}), \tag{243}$$

$$\beta_3 = \frac{N\lambda}{2\mu A}(B - \sqrt{B^2-4\mu A}), \tag{244}$$

where

$$A = N\lambda + \mu, \qquad B = N\lambda + \mu + \nu + \eta. \tag{245}$$

Thus, the general solution to Equations (238) is

$$p_n = C_1 + C_2\beta_2^n + C_3\beta_3^n, \qquad n=0,1,2,\cdots,S+1; \tag{246}$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting Equations (246) into (228,230,235) and evaluating for the q_n, we find

$$\left.\begin{aligned} q_o &= \eta^{-1}[C_1(N\lambda+\nu-\mu) + C_2(N\lambda+\nu-\mu\beta_2) + C_3(N\lambda+\nu-\mu\beta_3)]; \\ q_n &= \eta^{-1}\{C_1\nu + C_2\beta_2^{n-1}[-\mu\beta_2^2+(N\lambda+\mu+\nu)\beta_2-N\lambda] \\ &\quad + C_3\beta_3^{n-1}[-\mu\beta_3^2 + (N\lambda+\mu+\nu)\beta_3-N\lambda]\}, \quad n=1,2,3,\cdots,S; \\ &\vdots \end{aligned}\right\} \tag{247}$$

$$q_{S+1} = \eta^{-1}[C_1(\mu+\nu-N\lambda) + C_2\beta_2^S(\mu\beta_2+\nu\beta_2-N\lambda) + C_3\beta_3^S(\mu\beta_3+\nu\beta_3-N\lambda)].$$

The general stationary solution of the model is thus given by Equations (246-247), with β_1 and β_2 defined by (243-244), where the unknown constants C_1, C_2, and C_3 are to be evaluated using the boundary conditions (231) and (233) together with the normalizing condition (234).

Using the familiar formula for the sum of terms of a geometric progression, we find

$$\sum_{n=0}^{S+1} p_n = \sum_{n=0}^{S+1} (C_1+C_2\beta_2^n+C_3\beta_3^n)$$

$$= C_1(S+2) + C_2(1-\beta_2)^{-1}(1-\beta_2^{S+2}) + C_3(1-\beta_3)^{-1}(1-\beta_3^{S+2}), \quad (248)$$

and

$$\sum_{n=0}^{S+1} q_n = \nu^{-1}[C_1(N\lambda+\nu-\mu) + C_2(N\lambda+\nu-\mu\beta_2) + C_3(N\lambda+\nu-\mu\beta_3)]$$

$$+ \sum_{n=1}^{S} \eta^{-1}\{C_1\nu + C_2\beta_2^{n-1}[-\mu\beta_2^2 + (N\lambda+\mu+\nu)\beta_2 - N\lambda]$$

$$+ C_3\beta_3^{n-1}[-\mu\beta_3^2 + (N\lambda+\mu+\nu)\beta_3 - N\lambda]\}$$

$$+ \eta^{-1}[C_1(\mu+\nu-N\lambda) + C_2\beta_2^S(\mu\beta_2+\nu\beta_2-N\lambda) + C_3\beta_3^S(\mu\beta_3+\nu\beta_3-N\lambda)]$$

$$= \eta^{-1}\{C_1(N\lambda+\nu-\mu) + C_2(N\lambda+\nu-\mu\beta_2) + C_3(N\lambda+\nu-\mu\beta_3)$$

$$+ C_1\nu S + C_2(1-\beta_2)^{-1}(1-\beta_2^S)[-\mu\beta_2^2 + (N\lambda+\mu+\nu)\beta_2 - N\lambda]$$

$$+ C_3(1-\beta_3)^{-1}(1-\beta_3^S)[-\mu\beta_3^2 + (N\lambda+\mu+\nu)\beta_3 - N\lambda]$$

$$+ C_1(\mu+\nu-N\lambda) + C_2\beta_2^S(\mu\beta_2+\nu\beta_2-N\lambda) + C_3\beta_3^S(\mu\beta_3+\nu\beta_3-N\lambda)\}$$

$$= \eta^{-1}\nu[C_1(S+2) + C_2(1-\beta_2)^{-1}(1-\beta_2^{S+2}) + C_3(1-\beta_3)^{-1}(1-\beta_3^{S+2})], \quad (249)$$

so that the normalizing condition (234) may be expressed as

$$1 = (1+\nu\eta^{-1})[C_1(S+2) + C_2(1-\beta_2)^{-1}(1-\beta_2^{S+2}) + C_3(1-\beta_3)^{-1}(1-\beta_3^{S+2})]. \quad (250)$$

The boundary condition (231) yields

$$0 = -(N\lambda+\eta)\eta^{-1}[C_1(N\lambda+\nu-\mu) + C_2(N\lambda+\nu-\mu\beta_2)$$

$$+ C_3(N\lambda+\nu-\mu\beta_3)] + \nu(C_1+C_2+C_3)$$

$$= -\eta^{-1}\{C_1[N\lambda(N\lambda+\nu-\mu+\eta) - \eta\mu] + C_2[N\lambda(N\lambda+\nu+\eta) - \mu\beta_2(N\lambda+\eta)]$$

$$+ C_3[N\lambda(N\lambda+\nu+\eta) - \mu\beta_3(N\lambda+\eta)]\}$$

$$= -\eta^{-1}\{C_1[N\lambda(N\lambda+\nu-\mu+\eta) - \eta\mu] + C_2(1-\beta_2)^{-1}[N\lambda(\nu+\eta) - \mu\eta\beta_2]$$

$$+ C_3(1-\beta_3)^{-1}[N\lambda(\nu+\eta) - \mu\eta\beta_3]\}; \quad (251)$$

and, similarly, the boundary condition (233) yields

$$0 = -\eta\eta^{-1}[C_1(\mu+\nu-N\lambda) + C_2\beta_2^S(\mu\beta_2+\nu\beta_2-N\lambda) + C_3\beta_3^S(\mu\beta_3+\nu\beta_3-N\lambda)]$$

$$+ N\lambda\eta^{-1}\{C_1\nu + C_2\beta_2^{S-1}[-\mu\beta_2^2 + (N\lambda+\mu+\nu)\beta_2 - N\lambda]$$

$$+ C_2\beta_3^{S-1}[-\mu\beta_3^2 + (N\lambda+\mu+\nu\beta_3 - N\lambda]\}$$

$$+ \nu[C_1 + C_2\beta_2^{S+1} + C_3\beta_3^{S+1}]$$

$$= \eta^{-1}\{C_1[N\lambda(\eta+\nu) - \eta\nu]$$

$$+ C_2\beta_2^{S-1}[-\mu(N\lambda+\eta)\beta_2^2 + N\lambda(N\lambda+\mu+\nu+\eta)\beta_2 - (N\lambda)^2]$$

$$+ C_3\beta_3^{S-1}[-\mu(N\lambda+\eta)\beta_3^2 + N\lambda(N\lambda+\mu+\nu+\eta)\beta_3 - (N\lambda)^2]\}$$

$$= C_1\eta^{-1}[N\lambda(\eta+\nu) - \eta\mu]; \tag{252}$$

where, in each case, the last step was accomplished with the aid of Equation (242). Solving (250-252) simultaneously, we find

$$C_1 = 0, \tag{253}$$

$$C_2 = \eta(\nu+\eta)^{-1}(1-\beta_2)[N\lambda(\nu+\eta) - \mu\eta\beta_3]$$

$$\cdot \{(1-\beta_2^{S+2})[N\lambda(\nu+\eta)-\mu\eta\beta_3] - (1-\beta_3^{S+2})[N\lambda(\nu+\eta)-\mu\eta\beta_2]\}^{-1}, \tag{254}$$

$$C_3 = -\eta(\nu+\eta)^{-1}(1-\beta_3)[N\eta(\nu+\eta)-\mu\eta\beta_2]$$

$$\cdot \{(1-\beta_2^{S+2})[N\lambda(\nu+\eta)-\mu\eta\beta_3] - (1-\beta_3^{S+2})[N\lambda(\nu+\eta)-\mu\eta\beta_2]\}^{-1}. \quad (255)$$

The complete solution is thus given by Equations (246-247) with β_2, β_3, C_1, C_2, and C_3 defined by (243-244,253-255).

Models Involving Faulty Repairs

Instant Failures

As an embellishment of the models of Taylor and Jackson [232] and Toft and Boothroyd [240], we introduce the notion of stochastically faulty repairs which can result in the failure of a repaired machine before it can be delivered to the work area. In the real world, this phenomenon might occur as a result of Yes-No quality control testing of single-operation repair jobs (e.g., repair by modular replacement). The inspection function, perhaps a time consuming process itself, is performed on each newly repaired unit, so inspection time can be regarded as being a component part of the repair time on a unit. Let us call this new type of failure an "instant failure."

The method of attack will be the usual scheme of considering the possible transitions in a small increment of time Δt. Noting that the resulting model will be analogous to a two-stage cyclic queue with self feedback at the service stage, reference is made to Finch's work on feedback in cyclic queues.* The reader may find it of interest to

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*Finch [57] has developed models for several types of feedback in single-server cyclic queues with exogenous arrivals/departures at compare the current model with Finch's "single service" feedback model.

Model Formulation

Consider first the Taylor and Jackson problem. We adopt all the usual assumptions for this system (see Chapter III) and suppose in addition that a newly repaired unit will instantly fail with probability α ($0 \le \alpha < 1$). If an instant failure occurs, the unit is immediately returned to the repair queue. (There is no change in the state of the system.) It is essential to the combinatorial scheme that instant failures be assumed to occur without regard to the state of the system and in a manner totally independent of the operating and servicing characteristics of the machines.

We adopt the notation used to present the Taylor and Jackson model in Chapter III and ask, what is the probability $P_n(t+\Delta t)$ that there are n failed machines in the system at time t in terms of the probabilities of there being various numbers of failed machines at time t? The key to answering this question is the observation that, since instant failures occur as independent events,

$$\Pr\left\{\begin{array}{l|l}\text{there is one service completion} & \text{n units in}\\ \text{in the short interval } (t,t+\Delta t) & \text{system at}\\ \text{and it is an instant failure} & \text{time t}\end{array}\right\} = [\mu_n \Delta t + O(\Delta t)](\alpha), \quad (256)$$

$$\Pr\left\{\begin{array}{l|l}\text{there is one service completion} & \text{n units in}\\ \text{in the short interval } (t,t+\Delta t) & \text{system at}\\ \text{and it is } \textit{not} \text{ an instant failure} & \text{time t}\end{array}\right\} = [\mu_n \Delta t + O(\Delta t)](1-\alpha), \quad (257)$$

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some of the stages. His equations are not applicable to the current problem since an attempt to set the exogenous-arrival terms to zero results in factors of the indeterminant form $0 \div 0$.

where $O(\Delta t)$ has been used to denote the inclusion of quantities of smaller magnitude than Δt. Thus,

$$P_o(t+\Delta t) = (1-N\lambda\Delta t)P_o(t) + (1-N\lambda\Delta t)[(1-\alpha)\mu\Delta t P_1(t) + O(\Delta t); \qquad (258)$$

$$\begin{aligned} P_n(t+\Delta t) = {} & (1-N\lambda\Delta t)(1-n\mu\Delta t)P_n(t) + (1-N\lambda\Delta t)(\alpha n\mu\Delta t)P_n(t) \\ & + (1-N\lambda\Delta t)[(1-\alpha)(n+1)\mu\Delta t]P_{n+1}(t) \\ & + (N\lambda\Delta t)[1-(n-1)\mu\Delta t]P_{n-1}(t) + O(\Delta t), \qquad (259) \\ & \qquad n=1,2,3,\cdots,c-1; \end{aligned}$$

$$\begin{aligned} P_c(t+\Delta t) = {} & (1-N\lambda\Delta t)(1-c\mu\Delta t)P_c(t) + (1-N\lambda\Delta t)(\alpha c\mu\Delta t)P_c(t) \\ & + (1-N\lambda\Delta t)[(1-\alpha)c\mu\Delta t]P_{c+1}(t) \\ & + (N\lambda\Delta t)[1-(c-1)\mu\Delta t]P_{c-1}(t) + O(\Delta t); \qquad (260) \end{aligned}$$

$$\begin{aligned} P_n(t+\Delta t) = {} & (1-N\lambda\Delta t)(1-c\mu\Delta t)P_n(t) + (1-N\lambda\Delta t)(\alpha c\mu\Delta t)P_n(t) \\ & + (1-N\lambda\Delta t)[(1-\alpha)c\mu\Delta t]P_{n+1}(t) \\ & + (N\lambda\Delta t)(1-c\mu\Delta t)P_{n-1}(t) + O(\Delta t), \qquad (261) \\ & \qquad n=c+1,c+2,c+3,\cdots,S; \end{aligned}$$

$$P_{S+1}(t+\Delta t) = (1-c\mu\Delta t)P_{S+1}(t) + (\alpha c\mu\Delta t)P_{S+1}(t)$$

$$+ (N\lambda\Delta t)(1-c\mu\Delta t)P_S(t) + O(\Delta t). \tag{262}$$

Model Solution

Comparing Equations (258-262) with Equations (5) reveals that we have an example of the general birth-and-death model in which

$$\left.\begin{array}{lll} \lambda_n = N\lambda, & \mu_n = (1-\alpha)n\mu, & n=0,1,2,\cdots,c; \\ \lambda_n = N\lambda, & \mu_n = (1-\alpha)c\mu, & n=c,c+1,c+2,\cdots,S; \\ \lambda_{S+1} = 0, & \mu_{S+1} = (1-\alpha)c\mu, & n=S+1. \end{array}\right\} \tag{263}$$

Accordingly, the dynamic and stationary equations of state may be obtained by substituting Equations (263) into (6,9). The stationary solution follows from Equations (10-11):

$$\left.\begin{array}{ll} p_n = \left[\dfrac{N\lambda}{(1-\alpha)\mu}\right]^n \dfrac{1}{n!} p_o, & n=1,2,3,\cdots,c; \\ p_n = \left[\dfrac{N\lambda}{(1-\alpha)\mu}\right]^n \dfrac{1}{c!c^{n-c}} p_o, & n=c,c+1,c+2,\cdots,S+1; \end{array}\right\} \tag{264}$$

where p_o is to be evaluated from the normalizing condition,

$$\sum_{n=0}^{S+1} p_n = 1. \tag{265}$$

The Analogue to Taylor and Jackson's Model

A comparison of Equations (263-264) and (40-41) permits an illuminating observation. The model just obtained is equivalent to Taylor and Jackson's model [232] with the mean service rate of a channel reduced by the factor $(1-\alpha)$. Does this compare with our expectation of the instant failure process? Yes, because, as will be demonstrated, the total time τ required to successfully restore a unit to service is a negative-exponentially distributed variate with mean $[(1-\alpha)\mu]^{-1}$.

The instant failure process may be thought of as a series of independent Bernoulli trials with probability $1 - \alpha$ of successful repair on each trial. The probability of a run of $k - 1$ failures followed by one success is

$$v_k = \alpha^{k-1}(1-\alpha), \quad k=1,2,3,\cdots. \tag{266}$$

The discrete distribution (266) is just the unconditional distribution of the number k of trials at repair that will be required to successfully restore a unit to service.

Now, *given* that k trials at repair were required to successfully restore a unit to service, we observe that the total service time τ is comprised of the sum of k mutually independent intervals τ_j $(j=1,2,3, \cdots,k)$, each of which follows the same negative exponential distribution with mean $1/\mu$. Thus, it follows (see, e.g., [53], p.10) that τ must be (conditionally) distributed according to the k*th* Erlang distribution,*

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*Also known variously as a Pearson Type III, chi-square with

$$f(\tau|k) = \frac{\mu^k \tau^{k-1} e^{-\mu\tau}}{(k-1)!}, \quad 0 \le \tau < \infty. \tag{267}$$

Accordingly, the unconditional probability distribution for τ is

$$\begin{aligned} g(\tau) &= \sum_{k=1}^{\infty} f(\tau|k) v_k \\ &= \sum_{k=1}^{\infty} \frac{\mu^k \tau^{k-1} e^{-\mu\tau}}{(k-1)!} \alpha^{k-1}(1-\alpha) \\ &= (1-\alpha)\mu e^{-\mu\tau} \sum_{k=1}^{\infty} \frac{(\alpha\mu\tau)^{k-1}}{(k-1)!} \\ &= (1-\alpha)\mu e^{-\mu\tau} e^{\alpha\mu\tau} \\ &= (1-\alpha)\mu e^{-(1-\alpha)\mu\tau}, \quad 0 \le \tau < \infty, \end{aligned} \tag{268}$$

which is indeed a negative exponential distribution with mean $[(1-\alpha)\mu]^{-1}$.

The Analogue to Toft and Boothroyd's Model

The instant-failures analogue to Toft and Boothroyd's model [240] can be similarly developed. In this case, the stationary probabilities q_n, of there being n failed machines in the system, are

$$q_n = q_o(p_n/p_o)/(1-\alpha)^n, \quad n=1,2,3,\cdots,N+S, \tag{269}$$

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even degrees of freedom, or gamma distribution. Additional properties of this distribution are described in Chapter V.

where the p_n $(n=0,1,2,\cdots,N+S)$ are defined by Equations (59-61) and where q_o is to be determined from the condition

$$\sum_{n=0}^{N+S} q_n = 1. \tag{270}$$

The Relationship Between Cyclic Queues and Repairman Models with Spares

Introduction

A description of the relationship between cyclic queues and repairman models with spares appeared in Chapter III. That discussion will now be amplified with a demonstration of equivalence between the cyclic queueing model of Koenigsberg and the repairman model with spares of Toft and Boothroyd. A qualitative consideration of the general usefulness of cyclic queueing theory in the solution of repairman problems with spares will follow.

Koenigsberg's Two-Stage Model

Koenigsberg's paper [139] on the two-stage cyclic queueing system with multichannel negative-exponential service at each stage was briefly described in Chapter III. His results are in fact an independent derivation of the repairman model with spares due originally to Toft and Boothroyd [240]. Equivalence of the two solutions can be demonstrated by an appropriate interpretation of Koenigsberg's notation.

The first stage of Koenigsberg's model (see Figure 9) is assumed to consist of M repair stations, each station being independently characterized by a negative-exponential service time distribution with mean service time $1/\mu_1$, and a queue of machines awaiting repair.

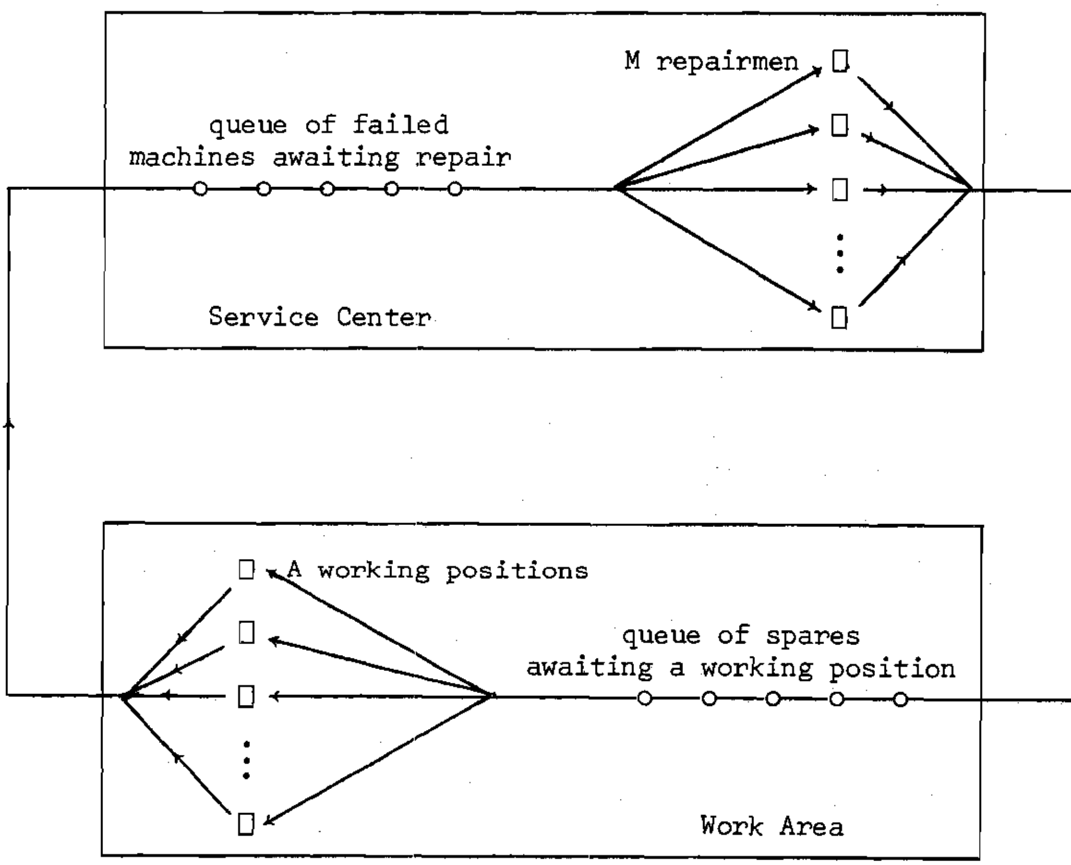

Figure 9. Koenigsberg's Two-Stage Cyclic Queue.

The second stage is assumed to consist of A working stations, each characterized by a negative-exponential working time distribution with mean working time $1/\mu_2$, and a queue of machines awaiting a work position. Thus, if there are N machines total in the cycle, the current state of the system can be specified as being the numbers of machines n_1, n_2 in stages 1 and 2, respectively, where $n_1 + n_2 = N$. Using $p(n_1,n_2)$ to denote the bivariate stationary probability that the system is in state (n_1,n_2), Koenigsberg obtained for $N - A \geq M$,

$$p(n_1,N-n_1) = \frac{A^{n_1}}{n_1!}\left(\frac{\mu_2}{\mu_1}\right)^{n_1} p(0,N), \qquad n_1=0,1,2,\cdots,M; \tag{271}$$

$$p(n_1,N-n_1) = \frac{A^{n_1}}{M!M^{n_1-M}}\left(\frac{\mu_2}{\mu_1}\right)^{n_1} p(0,N), \quad n_1=M,M+1,M+2,\cdots,N-A; \tag{272}$$

$$p(n_1,N-n_1) = \frac{A!A^{N-A}}{(N-n_1)!M!M^{n_1-M}}\left(\frac{\mu_2}{\mu_1}\right)^{n_1} p(0,N), \tag{273}$$

$$n_1=N-A,N-A+1,N-A+2,\cdots,N;$$

and for $N - A < M$,*

$$p(n_1,N-n_1) = \frac{A!}{n_1!}\left(\frac{\mu_2}{\mu_1}\right)^{n_1} p(0,N), \qquad n_1=0,1,2,\cdots,N-A; \tag{274}$$

$$p(n_1,N-n_1) = \frac{A!A^{N-A}}{(N-n_1)!n_1!}\left(\frac{\mu_2}{\mu_1}\right)^{n_1} p(0,N), \quad n_1=N-A,N-A+1,N-A+2,\cdots,M; \tag{275}$$

$$p(n_1,N-n_1) = \frac{A!A^{N-A}}{(N-n_1)!M!M^{n_1-M}}\left(\frac{\mu_2}{\mu_1}\right)^{n_1} p(0,N), \quad n_1=M,M+1,M+2,\cdots,N; \tag{276}$$

where in each case $p(0,N)$ is to be evaluated using the condition

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*A typographical error in Koenigsberg's equation (11) has been corrected in Equations (276). The intended form of Koenigsberg's equation (11) may be deduced from a comparison of it with his equation (13).

$$\sum_{n_1=0}^{N} p(n_1,N-n_1) = 1. \tag{277}$$

Making the notational substitutions,

$$\left.\begin{array}{lll} \mu_2 \rightarrow \lambda, & \mu_1 \rightarrow \mu, & \mu_2/\mu_1 \rightarrow \rho, \\ M \rightarrow c, & N-A \rightarrow S, & A \rightarrow N, \\ n_1 \rightarrow n, & \text{and} & p(n_1,N-n_1) \rightarrow p_n; \end{array}\right\} \tag{278}$$

we obtain from Equations (271-273) for $S \geq c$,

$$\left.\begin{array}{ll} p_n = (N^n \rho^n p_o)/n!, & n=0,1,2,\cdots,c; \\ p_n = (N^n \rho^n p_o)/(c!c^{n-c}), & n=c,c+1,c+2,\cdots,S; \\ p_n = (N!N^S \rho^n p_o)/[(N+S-n)!c!c^{n-c}], & n=S,S+1,S+2,\cdots,N+S; \end{array}\right\} \tag{279}$$

and from Equations (274-276) for $S < c$,

$$\left.\begin{array}{ll} p_n = (N!\rho^n p_o)/n!, & n=0,1,2,\cdots,S; \\ p_n = (N!N^S \rho^n p_o)/[(S+N-n)!n!], & n=S,S+1,S+2,\cdots,c; \\ p_n = (N!N^S \rho^n p_o)/[(S+N-n)!c!c^{n-c}], & n=c,c+1,c+2,\cdots,N+S. \end{array}\right\} \tag{280}$$

Equations (279-280) are seen to be identical to Equations (59-60) which are cited for Toft and Boothroyd [240] in Chapter III.

Some Observations

It is clear from the above example that the cyclic queueing approach (e.g., Koenigsberg's approach) to repairman problems with spares involves much the same considerations as the "direct" approach (e.g., Toft and Boothroyd's or Taylor and Jackson's approach).* However, it is apparent that even for a two-stage system, the cyclic queueing approach requires additional (yet redundant) notation and effort in formulation and solution that will not generally be necessary nor desirable in the study of repairman problems with spares. Thus, it would seem that the cyclic queueing approach will not be preferable to the "direct" approach except possibly in those instances where the introduction of special phenomena (e.g., phase-type service, unusual queue disciplines, etc.) necessitates the inclusion of additional variables in even the "direct" approach.

On the other hand, there can be no doubt as to the value of using *existing* cyclic queueing models** as shortcuts to finding solutions to repairman problems with spares whenever possible. Using the rationale illustrated with the Koenigsberg example, it is a fairly simple matter to manipulate them into the form of repairman models with spares (as is done in several later sections of this chapter). Further, it is

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*Unfortunately, due to the limited scope of the literature in both areas, the given example is the only case for which a direct comparison can be made.

**See specific references cited in the "Cyclic Queues" section of Chapter III.

clear that our knowledge of repairman models with spares will benefit from future advances in cyclic queueing theory and, to a lesser extent, from advances in finite and general queueing network theory.*

The extent of such potential benefit should not be overestimated. Many of the existing cyclic and finite queueing models did not prove useful in the present investigation. There were three primary reasons for this: When evaluated for the case of only *two* stages in *cyclic* order (i.e., the base case for repairman models with spares), most such models had to be discarded because they either (1) became identical to Koenigsberg's model [139] and hence duplicated Toft and Boothroyd's solution [240]; (2) reduced to a trivial model involving two single-server stages (one repairman and one working position) and hence duplicated portions of Barlow's work [5]; or (3) included phenomena (e.g., exogeneous arrivals) inconsistent with the conceptualization of a repairman system with spares.

Models with Transit Delays

Model Characteristics

We now consider the repairman problem with spares which includes the additional ramification of stochastic time delays in transporting units to and from the service center. The method of attack will involve the adaptation of certain results reported in the literature on cyclic and finite queues.

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*It is significant in this context to note that several new papers on cyclic and finite queues were published during the writing of this dissertation, while no new papers on repairman models with spares have appeared for several years.

More specifically, we seek to extend the model of Toft and Boothroyd [240] to include stochastically distributed transit delay times. The machine failure and service times are assumed to be negative exponential variates. Let λ be the mean failure rate and $1/\mu$ be the mean service time of an individual machine. Suppose that there are c service channels (repairmen), N working positions for machines, S spare machines, and a total of $M = N + S$ machines in the system. Let $(n,m;a,b)$ denote the state of the system when there are n machines in the work area (those working plus the spares "queueing" for a work position), m machines in the service center (those undergoing or awaiting repair), a failed machines in transit to the service center, and b repaired machines in transit to the work area. Let $p(n,m;a,b)$ denote the stationary probability of the system being in state $(n,m;a,b)$. Note that $M = n + m + a + b$.

To be consistent with the original model [240], we require that the failure and servicing phenomena operate in the usual way, specifically independent of the behavior of units in transit. That is, the (combined) mean rate at which machines fail, λ_n, depends only upon the number of machines working and not upon the values of m, a, or b. Similarly, the (combined) mean rate at which machines are being repaired, μ_m, depends only upon the number of machines in the service center and not upon the values of n, a, or b. We also assume that the transportation of units to the service center is an operation independent of the transportation of units to the work area, and that both are independent of the failure and servicing operations.

Finally, we specify the λ_n and μ_m for the case of interest:

$$\lambda_o \doteq 1, \qquad n=0; \tag{281}$$

$$\lambda_n = n\lambda, \qquad n=1,2,3,\cdots,N; \tag{282}$$

$$\lambda_n = N\lambda, \qquad n=N,N+1,N+2,\cdots,M; \tag{283}$$

$$\mu_o \doteq 1, \qquad m=0; \tag{284}$$

$$\mu_m = m\mu, \qquad m=1,2,3,\cdots,c; \tag{285}$$

$$\mu_m = c\mu, \qquad m=c,c+1,c+2,\cdots,M. \tag{286}$$

It is clear that in fact $\lambda_o = \mu_o \equiv 0$. The above values (281,284) are merely a notational convenience that will allow us to avoid writing extra equations for the special cases when n = 0 and/or m = 0.

Negative Exponential Transit Times

Conceptualization. Suppose that the transfer time of a failed unit to the service center is a negative-exponential variate with mean $1/\alpha$ and that the transfer time of a repaired unit to the work area is a negative-exponential variate with mean $1/\beta$. Then, conceptually, the desired model is equivalent to a four-stage cyclic queue with zero transit times and multichannel negative-exponential servicing at each stage. Let $\gamma_{i,j}$ be the mean service rate at stage i when there are j units queueing or being serviced at stage i. The four stages may then be characterized as follows:

Stage 1. The first stage consists of N working positions (servers) and $\gamma_{1,j} \equiv \lambda_j$ ($j=0,1,2,\cdots,M$). The queue at this stage contains spare machines waiting for a work position.

Stage 2. The second stage has M transportation devices (servers) and $\gamma_{2,j} = j\alpha$ ($j=0,1,2,\cdots,M$). There is no queue at this stage since there are sufficient servers to immediately accommodate each new arrival.

Stage 3. The third stage consists of c servers and $\gamma_{3,j} \equiv \mu_j$ ($j=0,1,2,\cdots,M$). The queue at this stage consists of failed machines awaiting service.

Stage 4. The fourth stage has M transportation devices (servers) and $\gamma_{4,j} = j\beta$ ($j=0,1,2,\cdots,M$). There is no queue at this stage since there are sufficient servers to immediately accommodate each new arrival.

If the second and fourth stages are eliminated, the system reduces to Koenigsberg's two-stage cyclic queue [139] for which equivalence to the Toft and Boothroyd model [240] was demonstrated earlier in this chapter. Further, it can be seen that the "immediate servicing" second and fourth stages provide precisely the desired transit delays.

Steady-State Equations. To obtain the equations governing the system, we relate the probability $P(n,m;a,b;t+\Delta t)$ of the system being in state $(n,m;a,b)$ at time $t + \Delta t$ to various probabilities of the system being in other states at time t by considering the possible transitions that can occur in a small increment of time Δt. Operating on these equations in the usual way, we obtain first the dynamic and then the steady-state equations of the system. It will not be productive to go through this procedure since we can more easily obtain the steady-state equations *and* the general form of their solution from Pelczynski's work [177].

For the finite queue with R interconnected stations and with negative-exponential serving at each station, Pelczynski gave the equations

$$\left[\sum_{i=1}^{R} \xi_i(n_i)\eta_i(n_i)\mu_i\right] P(n_1,n_2,\cdots,n_R)$$

$$= \sum_{i=1}^{R} \xi_i(n_i)\left[\sum_{j=1}^{R} p_{ji}\eta_j(n_j+1)\mu_j P(n_1,n_2,\cdots,n_i-1,\cdots,n_j+1,\cdots,n_R)\right], \qquad (287)$$

where $P(n_1,\cdots,n_R)$ is the probability that the system is in state $(n_1,\cdots,n_R)$ and where:

n_i = number of customers at service point i;

μ_i = mean service rate at a counter at service point i;

p_{ij} = probability that a customer leaving service point i will be sent to service point j;

m_i = number of parallel counters at service point i;

M = number of customers in the system;

$$\xi_i(0) = 0; \qquad \xi_i(n_i) = 1, \quad n_i \neq 0; \qquad (288)$$

$$\eta_i(0) = 1; \qquad \eta_i(n_i) = n_i, \quad 0<n_i\le m_i; \quad \eta_i(n_i) = m_i, \quad m_i<n_i\le M; \qquad (289)$$

$$p_{ii} = 0; \qquad \sum_{j=1}^{R} p_{ij} = 1. \tag{290}$$

We need not be concerned with Pelczynski's choice of $\eta_i(0) = 1$. This is a (commonly used) notational convenience that has no actual effect on the solution.

Temporarily adopting Pelczynski's notation for the case R = 4 and letting i denote the *ith* of the four cyclic stations defined above ($n \to n_1$, $m \to n_3$, $a \to n_2$, $b \to n_4$), we have for our system

$$\mu_1 = \lambda, \qquad \mu_2 = \alpha, \qquad \mu_3 = \mu, \qquad \mu_4 = \beta; \tag{291}$$

$$m_1 = N, \qquad m_2 = M, \qquad m_3 = c, \qquad m_4 = M; \tag{292}$$

$$p_{12} = p_{23} = p_{34} = p_{41} = 1 \quad \text{and} \quad p_{ij} = 0 \ \text{otherwise}. \tag{293}$$

<u>Stationary Solution</u>. Pelczynski found that the $P(n_1,\cdots,n_R)$ of Equation (287) had the general form

$$P(n_1,\cdots,n_R) = \frac{\left[\prod_{i=0}^{M} \eta_1(i)\right]\left[\prod_{i=2}^{R} \left(\frac{a_i\mu_1}{\mu_i}\right)^{n_i}\right] P(M,0,\cdots,0)}{\left[\prod_{i=0}^{n_1} \eta_1(i)\right]\left\{\prod_{i=2}^{R}\left[\prod_{j=0}^{n_1} \eta_i(j)\right]\right\}}, \tag{294}$$

where the coefficients $\{a_i\}$ are obtained by solving the equation

$$\begin{bmatrix} 1 & -p_{32} & -p_{42} & \cdots & -p_{R2} \\ -p_{23} & 1 & -p_{43} & \cdots & -p_{R3} \\ -p_{24} & -p_{34} & 1 & \cdots & -p_{R4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_{2R} & -p_{3R} & -p_{4R} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_R \end{bmatrix} = \begin{bmatrix} p_{12} \\ p_{13} \\ p_{14} \\ \vdots \\ p_{1R} \end{bmatrix}. \tag{295}$$

Substituting Equations (293) into (295) with R = 4, we have

$$\alpha_2 = 1; \tag{296}$$

$$-\alpha_2 + \alpha_3 = 0; \tag{297}$$

$$-\alpha_3 + \alpha_4 = 0; \tag{298}$$

from which it follows that

$$\alpha_2 = \alpha_3 = \alpha_4 = 1. \tag{299}$$

Substituting Equations (291,299) into (294) with R = 4 and using variously (289) and (292), we find

$$P(n_1,n_2,n_3,n_4) = \frac{\left[\prod_{i=0}^{M} \eta_1(i)\right]\left[\left(\frac{\lambda}{\alpha}\right)^{n_2}\left(\frac{\lambda}{\mu}\right)^{n_3}\left(\frac{\lambda}{\beta}\right)^{n_4}\right]P(M,0,0,0)}{\left[\prod_{i=0}^{n_1} \eta_1(i)\right]\left[\prod_{i=0}^{n_2} \eta_2(i)\right]\left[\prod_{i=0}^{n_3} \eta_3(i)\right]\left[\prod_{i=0}^{n_4} \eta_4(i)\right]}$$

$$= \frac{\lambda^M P(M,0,0,0)\left[\prod_{i=0}^{M} \eta_1(i)\right]}{n_2!n_4!\lambda^{n_1}\alpha^{n_2}\mu^{n_3}\beta^{n_4}\left[\prod_{i=0}^{n_1} \eta_1(i)\right]\left[\prod_{i=0}^{n_3} \eta_3(i)\right]} . \qquad (300)$$

Reverting to the original notation, that is, letting

$$\left.\begin{array}{c} P(n_1,n_2,n_3,n_4) \rightarrow p(n,m;a,b), \\ \\ n_1 \rightarrow n, \quad n_2 \rightarrow a, \quad n_3 \rightarrow m, \quad n_4 \rightarrow b; \end{array}\right\} \qquad (301)$$

and using Equations (281-286,289,292), we can write (300) in the form

$$p(n,m;a,b) = \frac{p(M,0;0,0)\left[\prod_{i=0}^{M} \lambda_i\right]}{a!b!\alpha^a\beta^b\left[\prod_{i=0}^{n} \lambda_i\right]\left[\prod_{i=0}^{m} \mu_i\right]} . \qquad (302)$$

Noting that the numerator in (302) is a constant which depends only on the system parameters and which is to be evaluated using the normalizing condition

$$\sum_{n=0}^{M} \sum_{m=0}^{M-n} \sum_{a=0}^{M-n-m} p(n,m;a,b) = 1, \quad \text{with} \tag{303}$$

$$b = M - n - m - a; \tag{304}$$

we can replace it with an also unknown constant term C_o without any loss of validity in representation. Doing so, we obtain

$$p(n,m;a,b) = C_o B_1(n) B_2(m)/[a!b!\alpha^a \beta^b], \tag{305}$$

where (for reasons to be made clear in the next section) we have made the additional substitutions

$$B_1(n) = \left[\prod_{i=0}^{n} \lambda_i\right]^{-1}, \tag{306}$$

$$B_2(m) = \left[\prod_{i=0}^{m} \mu_i\right]^{-1}. \tag{307}$$

As a partial check on our calculations, we might note that for the trivial case $N = c = 1$, Equation (305) agrees with a result deducible from Benson and Gregory's work [14].*

To evaluate C_o, we substitute (305) into (303) to find

---

*Benson and Gregory studied a problem involving negative-exponentially distributed transit times between stages of a k-stage cyclic queue. However, their results were limited to the case of single-server stages and were complicated by the inclusion of exogenous arrivals at each stage.

$$1 = \sum_{n=0}^{M} \sum_{m=0}^{M-n} \sum_{a=0}^{M-n-m} [C_o B_1(n) B_2(m)/(a!b!\alpha^a \beta^b)]$$

$$= C_o \sum_{n=0}^{M} \sum_{m=0}^{M-n} B_1(n) B_2(m) \sum_{a=0}^{M-n-m} \frac{(1/\alpha)^a (1/\beta)^{M-n-m-a}}{a!(M-n-m-a)!}$$

$$= C_o \sum_{n=0}^{M} \sum_{m=0}^{M-n} \frac{B_1(n) B_2(m)}{(M-n-m)!} \sum_{a=0}^{M-n-m} \binom{M-n-m}{a} \left(\frac{1}{\alpha}\right)^a \left(\frac{1}{\beta}\right)^{M-n-m-a}$$

$$= C_o \sum_{n=0}^{M} \sum_{m=0}^{M-n} \frac{B_1(n) B_2(m)}{(M-n-m)!} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^{M-n-m} . \tag{308}$$

The last step was accomplished with the aid of the binomial identity,

$$\sum_{i=0}^{K} \binom{K}{i} x^i y^{K-i} = (x+y)^K . \tag{309}$$

Hence, we have the simpler form for C_o,

$$C_o = \left[\sum_{n=0}^{M} \sum_{m=0}^{M-n} B_1(n) B_2(m) (\alpha^{-1} + \beta^{-1})^{M-n-m}/(M-n-m)! \right]^{-1} . \tag{310}$$

<u>Marginal Distribution of Units at Stages 1 and 3.</u> Let $p_{n,m}$ denote the marginal joint distribution of there being n units in the working area and m units in the service center. Then,

$$p_{n,m} = \sum_{a=0}^{M-n-m} p(n,m;a,b)$$

$$= \sum_{a=0}^{M-n-m} C_o B_1(n) B_2(m)/[a!b!\alpha^a \beta^b]$$

$$= \frac{C_o B_1(n) B_2(m)}{(M-n-m)!} \sum_{a=0}^{M-n-m} \binom{M-n-m}{a} \left(\frac{1}{\alpha}\right)^a \left(\frac{1}{\beta}\right)^{M-n-m-a}$$

$$= C_o B_1(n) B_2(m) (\alpha^{-1}+\beta^{-1})^{M-n-m}/(M-n-m)!, \tag{311}$$

where again the binomial identity has been used. It is interesting to note that the distribution depends on the *total* delay time of a unit in one complete circuit of the system and not on the individual delay times at each of stages 2 and 4.

Conditional Distribution of Units in Transit. Let $q(a|n,m)$ denote the conditional probability of there being a units in transit to the service center (and hence M-n-m-a units in transit to the work area). Then,

$$q(a|n,m) = p(n,m;a,b)/p_{n,m}$$

$$= \frac{C_o B_1(n) B_2(m)/[a!b!\alpha^a \beta^b]}{C_o B_1(n) B_2(m)(\alpha^{-1}+\beta^{-1})^{M-n-m}/(M-n-m)!}$$

$$= \binom{M-n-m}{a} \frac{(\alpha^{-1})^a (\beta^{-1})^{M-n-m-a}}{(\alpha^{-1}+\beta^{-1})^{M-n-m}} . \tag{312}$$

Further, the expected value of a for given n,m is

$$\bar{a} = \sum_{a=0}^{M-n-m} a q(a|n,m)$$

$$= [(\alpha^{-1}+\beta^{-1})^{M-n-m}]^{-1} \sum_{a=0}^{M-n-m} \binom{M-n-m}{a} a(\alpha^{-1})^a (\beta^{-1})^{M-n-m-a}. \tag{313}$$

To evaluate the sum in (313), we first observe that

$$\sum_{i=0}^{K} \binom{K}{i} i x^i y^{K-i} = x \sum_{i=1}^{K} \binom{K}{i} i x^{i-1} y^{K-i}$$

$$= x \frac{d}{dx}\left[\sum_{i=1}^{K} \binom{K}{i} x^i y^{K-i}\right]$$

$$= x \frac{d}{dx} [(x+y)^K - y^K]$$

$$= Kx(x+y)^{K-1}, \tag{314}$$

when use is made of the binomial identity (309). Applying (314) to (313), we find

$$\bar{a} = [(\alpha^{-1}+\beta^{-1})^{M-n-m}]^{-1}(M-n-m)(1/\alpha)(\alpha^{-1}+\beta^{-1})^{M-n-m-1}, \tag{315}$$

or

$$\bar{a} = (M-n-m)\alpha^{-1}/(\alpha^{-1}+\beta^{-1}). \tag{316}$$

By symmetry, or using (304), we have also

$$\bar{b} = (M-n-m)\beta^{-1}/(\alpha^{-1}+\beta^{-1}). \tag{317}$$

Equations (316-317) should have been anticipated on intuitive grounds.

Generally Distributed Transit Times

In this subsection, we indicate how problems involving more generally distributed transit delays may be approached. Retain the meaning of the symbols n, m, a, b, N, c, M as defined in general for this section on transit delays. Identify stage 1 as the working area with mean failure rates λ_n given by Equations (281-283). Identify stage 2 as the service center with mean repair rates μ_m given by Equations (284-286). Let $g_1(x)$ be the probability density function, $G_1(x)$ be the cumulative distribution function, and H_1 be the average value of the transit time x from the working area to the service center. Let $g_2(y)$, $G_2(y)$, and H_2 be the analogous quantities for the transit time y from the service center to the working area. Finally, let $\vec{x}_a$ and $\vec{y}_b$ be vector abbreviations for the elapsed transit times $x_1, x_2, x_3, \cdots, x_a$ and $y_1, y_2, y_3, \cdots, y_b$ of the a units traveling from stage 1 to stage 2 and of the b units traveling from stage 2 to stage 1, respectively. Then, adapting a result obtained by Posner and Bernholtz [180], we can write the stationary state-probabilities in the form

$$P_{n,m;a,b}(\vec{x}_a,\vec{x}_b) = CB_1(n)B_2(m)\left\{\prod_{i=0}^{a} [1-G_1(x_i)]\right\}\left\{\prod_{i=0}^{b} [1-G_2(y_i)]\right\}, \qquad (318)$$

where $B_1(n)$ and $B_2(m)$ are again defined by Equations (306-307); where

$$G_1(x_o) \doteq 0, \qquad G_2(y_o) \doteq 0 \qquad (319)$$

as a notational convenience; and where the normalizing constant C is given by

$$C = \left[\sum_{n=0}^{M} \sum_{m=0}^{M-n} B_1(n)B_2(m)(H_1+H_2)^{M-n-m}/(M-n-m)!\right]^{-1}. \qquad (320)$$

For the marginal joint probability of n machines at stage 1 and m machines at stage 2, it follows from Posner and Bernholtz that

$$p_{n,m} = CB_1(n)B_2(m)(H_1+H_2)^{M-n-m}/(M-n-m)!. \qquad (321)$$

A Problem in Servicing Aircraft Engines

Description

The system in which aircraft engines are maintained by the commercial airlines is shown in Figure 10.

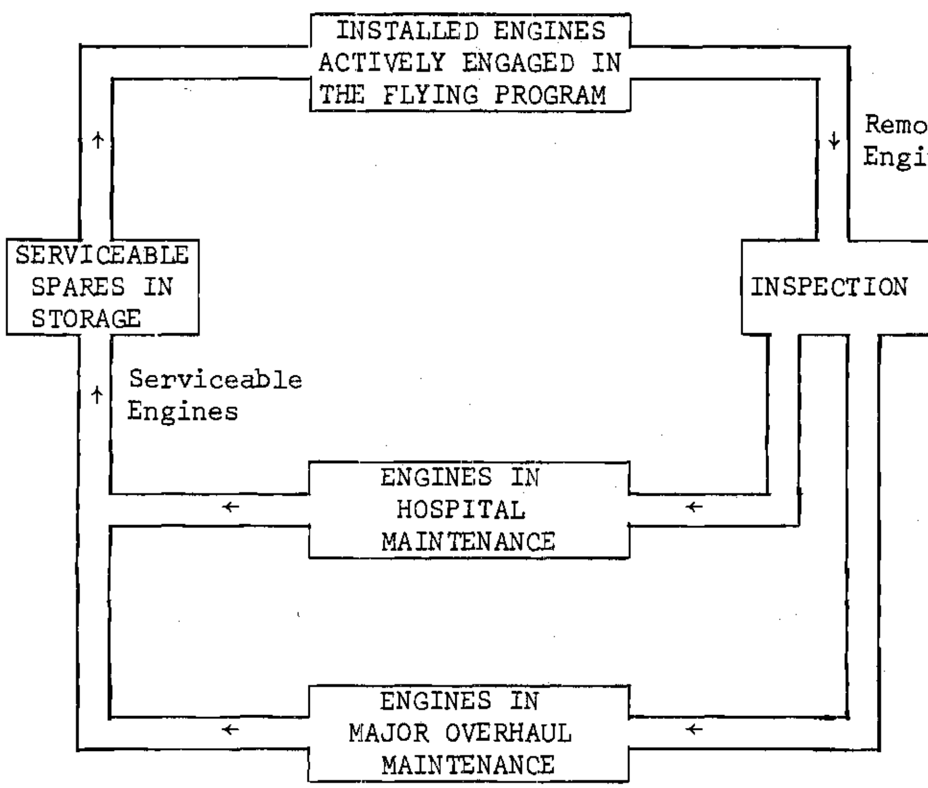

Figure 10. The Commercial Airlines' "Engine Pipeline"

The most distinguishing characteristic of system flow is the "maximum time rule" for operation of an engine. Government regulations require that engines not be operated a time longer than a specified, fixed maximum time T since their last major overhaul. (See Figure 11.)

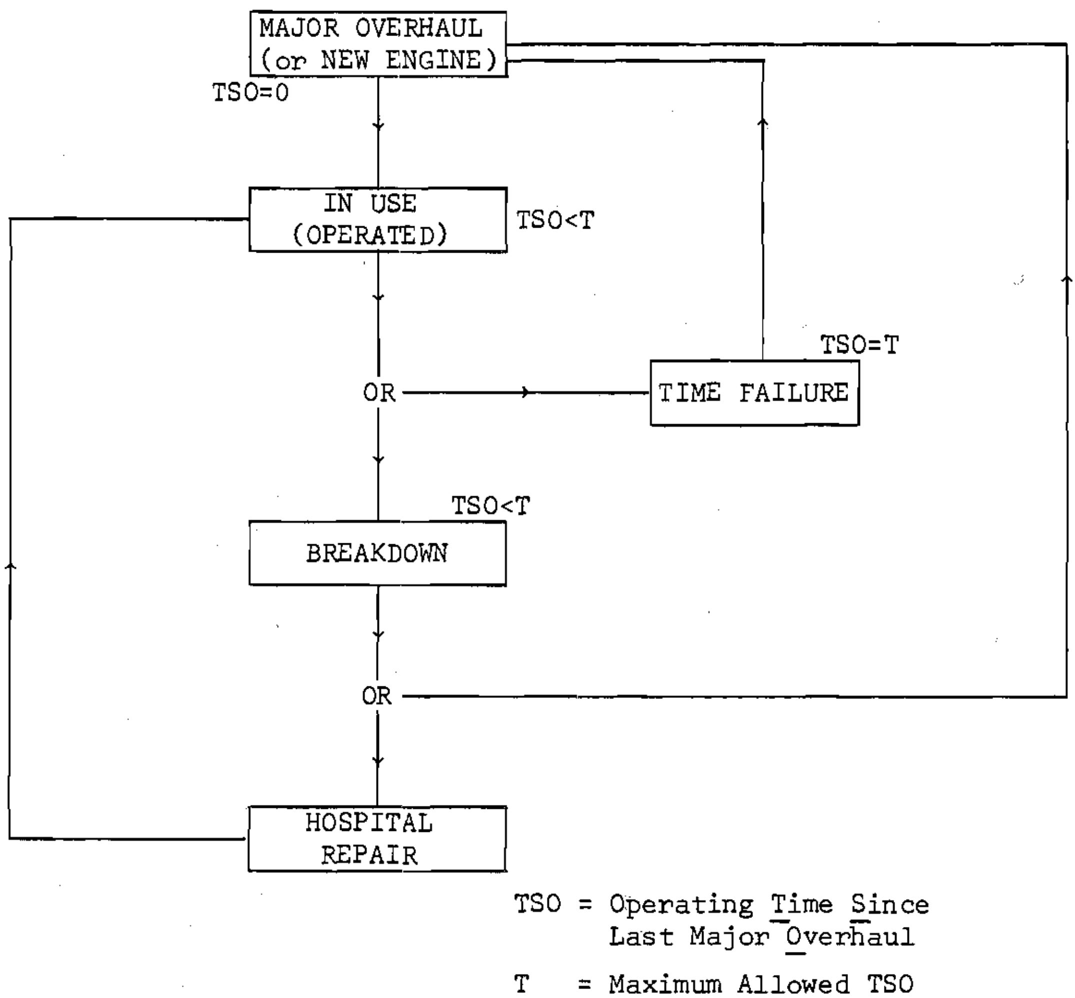

Figure 11. Measurement of Aircraft Engine Age

Thus, the failure characteristics of an engine are generally specified in terms of the time since overhaul (TSO). It is to be expected that the failure characteristics of engines will be a function of their total maintenance history.* However, as Hines and Moder [85] have shown, meaningful failure-time density functions for overhaul removals and hospital removals can be based on the current TSO alone.**

The Model

In order to investigate this problem with our current analytical resources, it will be necessary to make several simplifying assumptions. First, we assume that the system can be reticulated into three independent sections between which instantaneous transfers take place, as shown in Figure 12. Second, we assume that the time to failure of an installed engine depends only on the length of time it has been installed and not on the last type of service received. Third, we assume that the (instantaneous) inspection operation results in sending a fraction p ($0<p<1$) of the failed machines to station 2 (hospital maintenance) and the remainder to station 3 (overhaul maintenance). Finally, we assume that operations at each of the three stations are such that each may be characterized as a $M|M|c$ queueing system.

Suppose there are N operating positions for machines and a total of M ($M \geq N$) machines in the system. Let n_j ($j=1,2,3$) denote the number of units in station j and let $p(n_1,n_2,n_3)$ be the stationary probability of there being n_1, n_2, n_3 units in stations 1, 2, 3, respectively.

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*It is, in fact, standard practice in the commercial airline industry to keep records of *all* maintenance performed on each engine.

**See Hines and Moder's paper [85] for a more thorough discussion of the aircraft engine maintenance problem.

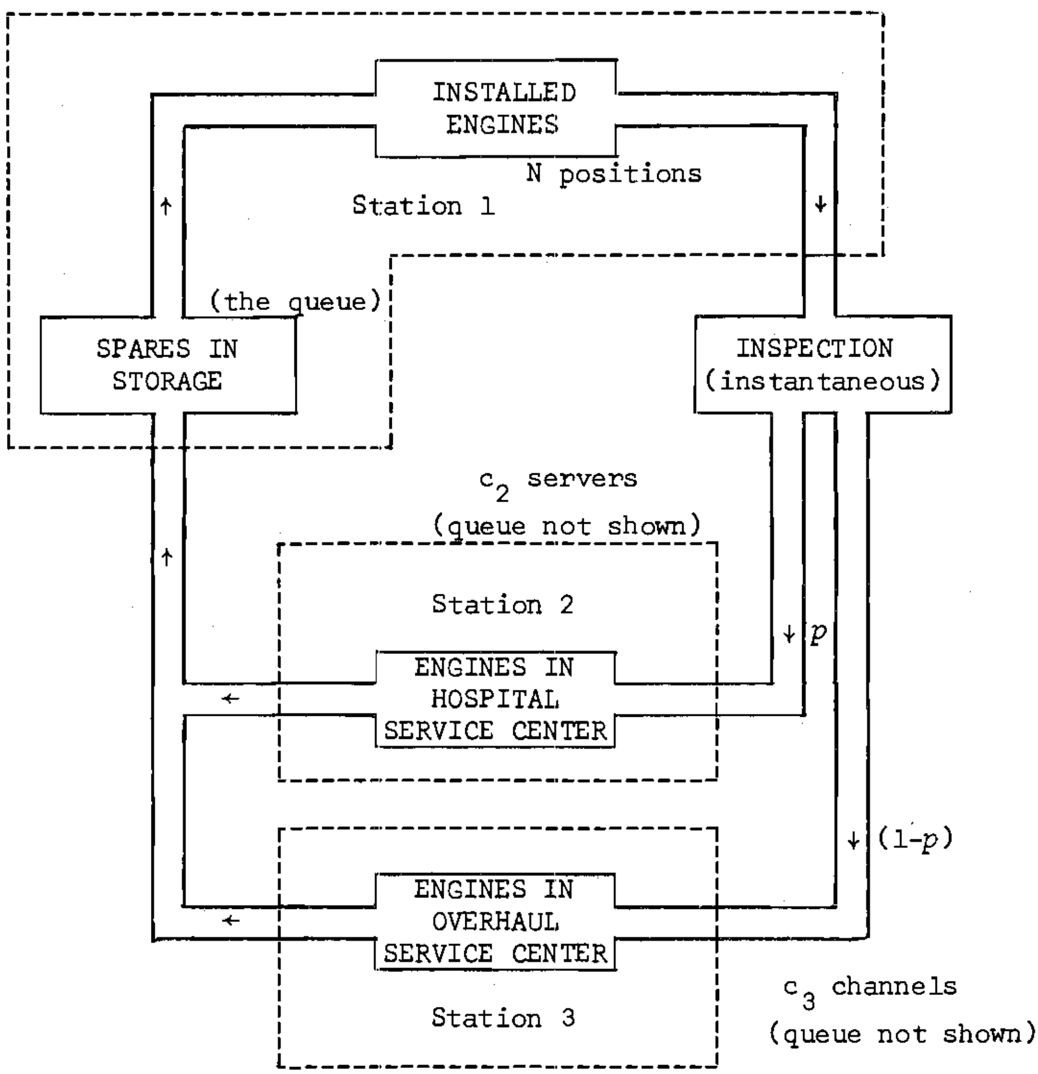

Figure 12. Finite-Queue Model for Aircraft Engine Problem

Let λ be the mean failure rate of an individual machine. Let α_j ($j=2,3$) be the mean service rate of an individual channel and c_j the number of channels at service center j. Denote by λ_{n_1}, α_{2,n_2}, α_{3,n_3}, the mean "service" rates when there are n_1, n_2, n_3 at stations 1, 2, 3, respectively.

The mean service rates are

$$\lambda_o \doteq 1, \qquad n_1=0; \tag{322}$$

$$\lambda_{n_1} = n_1\lambda, \qquad n_1=1,2,3,\cdots,N; \tag{323}$$

$$\lambda_{n_1} = N\lambda, \qquad n_1=N,N+1,N+2,\cdots,M; \tag{324}$$

$$\alpha_{2,o} \doteq 1, \qquad n_2=0; \tag{325}$$

$$\alpha_{2,n_2} = n_2\alpha_2, \qquad n_2=1,2,3,\cdots,c_2; \tag{326}$$

$$\alpha_{2,n_2} = c_2\alpha_2, \qquad n_2=c_2,c_2+1,c_2+2,\cdots,M; \tag{327}$$

$$\alpha_{3,o} \doteq 1, \qquad n_3=0 \tag{328}$$

$$\alpha_{3,n_3} = n_3\alpha_3, \qquad n_3=1,2,3,\cdots,c_3; \tag{329}$$

$$\alpha_{3,n_3} = c_3\alpha_3, \qquad n_3=c_3,c_3+1,c_3+2,\cdots,M; \tag{330}$$

where the convention $\lambda_o = \alpha_{2,o} = \alpha_{3,o} \doteq 1$ is a notational convenience.

The model is thus a *finite queueing model* of the type studied by Pelczynski [177]. The steady state equations are given by (287) for $R = 3$ and

$$\mu_1 = \lambda, \qquad \mu_2 = \alpha_2, \qquad \mu_3 = \alpha_3; \tag{331}$$

$$m_1 = N, \qquad m_2 = c_2, \qquad m_3 = c_3; \tag{332}$$

$$[p_{ij}] = \begin{bmatrix} 0 & p & (1-p) \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{333}$$

Accordingly, Equation (295) takes the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p \\ 1-p \end{bmatrix}, \tag{334}$$

so that

$$a_2 = p, \qquad a_3 = 1 - p. \tag{335}$$

Substituting these values into Equation (294) yields the stationary solution

$$p(n_1,n_2,n_3) = \frac{\left[\prod_{i=0}^{M} \eta_i(i)\right]\left[\left(\frac{p\lambda}{\alpha_2}\right)^{n_2}\left(\frac{(1-p)\lambda}{\alpha_3}\right)^{n_3}\right] p(M,0,0)}{\left[\prod_{i=0}^{n_1} \eta_1(i)\right]\left[\prod_{i=0}^{n_2} \eta_2(i)\right]\left[\prod_{i=0}^{n_3} \eta_3(i)\right]}$$

$$= \frac{\left[\prod_{i=0}^{M} \lambda\eta_1(i)\right] p^{n_2}(1-p)^{n_3}\, p(M,0,0)}{\left[\prod_{i=0}^{n_1} \lambda\eta_1(i)\right]\left[\prod_{i=0}^{n_2} \alpha_2\eta_2(i)\right]\left[\prod_{i=0}^{n_3} \alpha_3\eta_3(i)\right]}$$

$$= \frac{\left[\prod_{i=0}^{M} \lambda_i\right] p^{n_2}(1-p)^{n_3}\; p(M,0,0)}{\left[\prod_{i=0}^{n_1} \lambda_i\right]\left[\prod_{i=0}^{n_2} \alpha_{2,i}\right]\left[\prod_{i=0}^{n_3} \alpha_{3,i}\right]}. \tag{336}$$

Writing

$$B_1(n_1) = \left[\prod_{i=0}^{n_1} \frac{1}{\lambda_i}\right]^{-1}; \tag{337}$$

$$B_j(n_j) = \left[\prod_{i=0}^{n_j} \frac{1}{\alpha_{j,i}}\right]^{-1}, \quad j=2,3; \tag{338}$$

and using a single constant C_o to replace all of the constant terms in (336), the solution can be written in the more compact form

$$p(n_1,n_2,n_3) = C_o p^{n_2}(1-p)^{n_3}\, B_1(n_1)B_2(n_2)B_3(n_3). \tag{339}$$

The constant C_o is to be obtained from the normalizing condition

$$\sum_{n_1=0}^{M} \sum_{n_2=0}^{M-n_1} p(n_1,n_2,M-n_1-n_2) = 1, \tag{340}$$

which reflects the equality

$$M = n_1 + n_2 + n_3. \tag{341}$$

CHAPTER V

MODELS WITH CHI-SQUARE SERVICE TIME DISTRIBUTIONS

Introduction

Models for two basic types of repairman systems with spares will be developed in this chapter. The models are characterized as having negative exponential machine failure-time distributions, scale-modified chi-square machine service-time distributions, and a single server. A later section of the chapter will consider (qualitatively) extension of the method employed to models having chi-square failure-time distributions, chi-square service-time distributions, and/or multiple servers.

Chi-Square Service Time Distributions

In Chapter IV, analysis was confined to problems with service times following the one-parameter (μ) negative exponential distribution. In the present chapter, we will considerably broaden the scope of potential applications of repairman models with spares by developing models with service times following a two-parameter (μ,J) probability distribution, *viz.*,

$$f_J(t) = \frac{(J\mu)^J}{(J-1)!}\, t^{J-1}\, e^{-J\mu t}, \quad 0 \le t < \infty, \tag{342}$$

where μ is a positive real constant and J is a positive integer. The distribution (342) is sometimes referred to as the "J*th* Erlang distribution." The substitution $\mu' = J\mu$ in (342) will result in the familiar chi-square distribution with even degrees of freedom. More specifically, $2\mu' t$ is chi-square distributed with $2J$ degrees of freedom. Accordingly, the distribution (342) may be regarded as being a scale-modified chi-square density function with even degrees of freedom.*

The set of functions $(J=1,2,3,\cdots)$ have a number of interesting properties. For example, in each case, the mean service time $\bar{t}_J$ is

$$\bar{t}_J = 1/\mu, \quad J=1,2,3,\cdots, \tag{343}$$

independent of J. The mode τ_J and variance V_J are

$$\tau_J = (J-1)/(\mu J), \quad J=1,2,3,\cdots; \tag{344}$$

and

$$V_J = 1/(J\mu^2), \qquad J=1,2,3,\cdots. \tag{345}$$

When $J = 1$, the function (342) reduces to the negative exponential distribution. When $J \to \infty$ in Equations (343-345), we have

$$\bar{t}_\infty = \tau_\infty = 1/\mu \quad \text{and} \quad v_\infty = 0, \tag{346}$$

which can be identified as the case of constant service times of length $1/\mu$.

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*It is also known variously as a Pearson Type III distribution or gamma distribution.

A very useful property of the distribution (342) stems from its relationship with the negative exponential distribution. It is a well-known fact (see, e.g., [53], p.10) that, if J independent random variables $t_1, t_2, t_3, \cdots, t_J$ each follow the same negative exponential distribution with mean $1/(J\mu)$, the sum $t = t_1 + t_2 + t_3 + \cdots + t_J$ has the distribution (342) with mean $1/\mu$. Thus, service times which follow the distribution (342) may be thought of as occurring in J successive, identical steps (phases), each step requiring a negative-exponentially distributed execution time with mean $1/(J\mu)$.

Two Model Types of Interest

As in previous chapters, we distinguish between two basic operating policies for repairman systems with spares. Let N denote the number of working positions for machines and let S denote the number of spare machines in the system. It is apparent that when S + 1 machines are in the service facility, it is no longer possible to operate the system at its maximum level of N working machines. There are accordingly two cases of interest:

Case I. All machines stop operating until servicing can restore the number of operative machines to N.

Case II. A number of machines less than N continues to operate.

In this chapter, we will refer to a model as being of Type I (Type II) if it incorporates a Case I (Case II) operating policy. In particular, it is noted that Taylor and Jackson's model [232] is Type I, while Toft and Boothroyd's model [240] is Type II.

The Type I Model with Chi-Square Servicing

Description of the Model

In this section, a model will be derived for the Type I repairman system with spares in which: (a) the machine failure times follow Equation (2), a negative exponential distribution with mean $1/\lambda$; (b) the machine service times follow Equation (342), a scale-modified chi-square distribution with 2J degrees of freedom and mean $1/\mu$; and (c) there are N working positions for machines, N + S machines total, and a single server.

Let S_n denote the state of the system; λ_n, the (combined) mean rate at which machines are failing; and μ_n, the mean rate at which a failed machine is being repaired when there are n $(n=0,1,2,\cdots,S+1)$ failed machines in the system. Let $P_n(t)$ and p_n denote the dynamic and stationary probabilities, respectively, that the system is in state S_n.

The λ_n and μ_n are the same as for Taylor and Jackson's single-server model, that is, as given by Equations (40) with c = 1. Unfortunately in this case, due to the complexity of the servicing phenomenon, it is clear that the process of state transitions $S_i \to S_j$ does not form a Markov chain. However, if we make use of one of the properties of the distribution (342) discussed above by considering a service to consist of J internal phases, each having a negative-exponentially distributed execution with mean $1/(J\mu)$, and partition each of the states S_n accordingly, we obtain an equivalent elementary Markov process.*

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*This scheme, the so-called "method of successive stages," is a commonly used device in investigations of queueing problems involving chi-square distributions. (See, e.g., [40], § 5.2; [159], *passim.*)

Define S_o, λ_o, and μ_o as before. Let $S_{n,j}$ denote the state of the system; $\lambda_{n,j}$, the (combined) mean rate at which machines are failing; and $\mu_{n,j}$, the mean rate at which a failed machine is being processed through the j*th* phase of service when there are n (n=1,2,3, $\cdots$,S+1) failed machines in the system *and* the unit being serviced is in phase j (j=1,2,3,$\cdots$,J) of that service. Then,

$$\left.\begin{aligned} \lambda_o &= N\lambda, & \mu_o &= 0, & & n=0; \\ \lambda_{n,j} &= N\lambda, & \mu_{n,j} &= J\mu, & & n=1,2,3,\cdots,S \text{ and } j=1,2,3,\cdots,J; \\ \lambda_{S+1,j} &= 0, & \mu_{S+1,j} &= J\mu, & & n=S+1 \text{ and } j=1,2,3,\cdots,J. \end{aligned}\right\} \qquad (347)$$

The process of transitions among the states S_o and the $S_{n,j}$ is seen to be a simple Markov process of the type studied previously. Further, since the transition times between states are all negative-exponential variates, we can follow the usual procedures for Poisson queues in formulating the equations of state.

Formulation of Equations

We use an approach analogous to that outlined in Chapter I for the general birth-and-death model. Attention focuses on the possible happenings in a small interval of time Δt. If the phases are numbered in reverse order of their execution so that a unit enters service at phase J and completes service at phase 1, the state probabilities at times t + Δt and t are related by

$$P_o(t+\Delta t) = (1-N\lambda\Delta t)P_o(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{1,1}(t) + O(\Delta t); \qquad (348)$$

$$P_{1,j}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{1,j}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{1,j+1}(t)$$

$$+ O(\Delta t), \quad j=1,2,3,\cdots,J-1; \qquad (349)$$

$$P_{1,J}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{1,J}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{2,1}(t)$$

$$+ (N\lambda\Delta t)P_o(t) + O(\Delta t); \qquad (350)$$

$$P_{n,j}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{n,j}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{n,j+1}(t)$$

$$+ (N\lambda\Delta t)(1-J\mu\Delta t)P_{n-1,j}(t) + O(\Delta t), \qquad (351)$$

$$n=2,3,4,\cdots,S \text{ and } j=1,2,3,\cdots,J-1;$$

$$P_{n,J}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{n,J}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{n+1,1}(t)$$

$$+ (N\lambda\Delta t)(1-J\mu\Delta t)P_{n-1,J}(t) + O(\Delta t), \qquad (352)$$

$$n=2,3,4,\cdots,S-1;$$

$$P_{S,J}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{S,J}(t) + (J\mu\Delta t)P_{S+1,1}(t)$$

$$+ (N\lambda\Delta t)(1-J\mu\Delta t)P_{S-1,J}(t) + O(\Delta t); \qquad (353)$$

$$P_{S+1,j}(t+\Delta t) = (1-J\mu\Delta t)P_{S+1,j}(t) + (J\mu\Delta t)P_{S+1,j+1}(t)$$

$$+ (N\lambda\Delta t)(1-J\mu\Delta t)P_{S,j}(t) + O(\Delta t), \quad (354)$$

$$j=1,2,3,\cdots,J-1;$$

$$P_{S+1,J}(t+\Delta t) = (1-J\mu\Delta t)P_{S+1,J}(t) + (N\lambda\Delta t)(1-J\mu\Delta t)P_{S,J}(t) + O(\Delta t); \quad (355)$$

where $O(\Delta t)$ denotes the inclusion of quantities of a smaller order of magnitude than Δt.

To obtain the dynamic equations of the system, we divide each of Equations (348-355) by Δt and take the limit as $\Delta t \to 0$. Recalling that

$$\lim_{\Delta t \to 0} \left\{ \frac{f(t+\Delta t)-f(t)}{\Delta t} \right\} = \frac{df(t)}{dt}, \quad (356)$$

we find

$$\frac{dP_o(t)}{dt} = -N\lambda P_o(t) + J\mu P_{1,1}(t); \quad (357)$$

$$\frac{dP_{1,j}(t)}{dt} = -(N\lambda+J\mu)P_{1,j}(t) + J\mu P_{1,j+1}(t), \quad j=1,2,3,\cdots,J-1; \quad (358)$$

$$\frac{dP_{1,J}(t)}{dt} = -(N\lambda+J\mu)P_{1,J}(t) + J\mu P_{2,1}(t) + N\lambda P_o(t); \quad (359)$$

$$\frac{dP_{n,j}(t)}{dt} = -(N\lambda+J\mu)P_{n,j}(t) + J\mu P_{n,j+1}(t) + N\lambda P_{n-1,j}(t), \tag{360}$$

$$n=2,3,4,\cdots,S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$\frac{dP_{n,J}(t)}{dt} = -(N\lambda+J\mu)P_{n,J}(t) + J\mu P_{n+1,1}(t) + N\lambda P_{n-1,J}(t), \tag{361}$$

$$n=2,3,4,\cdots,S;$$

$$\frac{dP_{S+1,j}(t)}{dt} = -J\mu P_{S+1,j}(t) + J\mu P_{S+1,j+1}(t) + N\lambda P_{S,j}(t), \tag{362}$$

$$j=1,2,3,\cdots,J-1;$$

$$\frac{dP_{S+1,J}(t)}{dt} = -J\mu P_{S+1,J}(t) + N\lambda P_{S,J}(t). \tag{363}$$

In the steady-state, Equations (357-363) become

$$0 = -N\lambda p_o + J\mu p_{1,1}; \tag{364}$$

$$0 = -(N\lambda+J\mu)p_{1,j} + J\mu p_{1,j+1}, \quad j=1,2,3,\cdots,J-1; \tag{365}$$

$$0 = -(N\lambda+J\mu)p_{1,J} + J\mu p_{2,1} + N\lambda p_o; \tag{366}$$

$$0 = -(N\lambda+J\mu)p_{n,j} + J\mu p_{n,j+1} + N\lambda p_{n-1,j}, \tag{367}$$

$$n=2,3,4,\cdots,S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$0 = -(N\lambda+J\mu)p_{n,J} + J\mu p_{n+1,1} + N\lambda p_{n-1,J}, \quad n=2,3,4,\cdots,S; \tag{368}$$

$$0 = -J\mu p_{S+1,j} + J\mu p_{S+1,j+1} + N\lambda p_{S,j}, \quad j=1,2,3,\cdots,J-1; \tag{369}$$

$$0 = -J\mu p_{S+1,J} + N\lambda p_{S,J}. \tag{370}$$

Making the substitution

$$\alpha \doteq \frac{N\lambda}{J\mu} \tag{371}$$

and rearranging terms, we can express these equations in the more convenient form

$$0 = p_{1,1} - \alpha p_o; \tag{372}$$

$$0 = p_{1,j+1} - (\alpha+1)p_{1,j}, \quad j=1,2,3,\cdots,J-1; \tag{373}$$

$$0 = p_{2,1} - (\alpha+1)p_{1,J} + \alpha p_o; \tag{374}$$

$$0 = p_{n,j+1} - (\alpha+1)p_{n,j} + \alpha p_{n-1,j}, \tag{375}$$

$$n=2,3,4,\cdots,S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$0 = p_{n+1,1} - (\alpha+1)p_{n,J} + \alpha p_{n-1,J}, \quad n=2,3,4,\cdots,S; \tag{376}$$

$$0 = p_{S+1,j+1} - p_{S+1,j} + \alpha p_{S,j}, \quad j=1,2,3,\cdots,J-1; \tag{377}$$

$$0 = p_{S+1,J} - \alpha p_{S,J}. \tag{378}$$

Before proceeding with the solution of Equations (372-378) for arbitrary J, it will be helpful to treat the case J = 2. This stopover will permit a more vivid disclosure of the general solution rationale. Further, within the limited scope of the J = 2 system, it will be possible to relate the results of the general-solution method to those obtained from a recursive approach.

Solution for the Case J = 2

When J = 2, $\alpha = N\lambda/(2\mu)$ and Equations (372-378) become

$$0 = p_{1,1} - \alpha p_o; \tag{379}$$

$$0 = p_{1,2} - (\alpha+1)p_{1,1}; \tag{380}$$

$$0 = p_{2,1} - (\alpha+1)p_{1,2} + \alpha p_o; \tag{381}$$

$$0 = p_{n,2} - (\alpha+1)p_{n,1} + \alpha p_{n-1,1}, \quad n=2,3,4,\cdots,S; \tag{382}$$

$$0 = p_{n+1,1} - (\alpha+1)p_{n,2} + \alpha p_{n-1,2}, \quad n=2,3,4,\cdots,S; \tag{383}$$

$$0 = p_{S+1,2} - p_{S+1,1} + \alpha p_{S,1}; \tag{384}$$

$$0 = p_{S+1,2} - \alpha p_{S,2}. \tag{385}$$

Recursive Approach. The most direct approach is a recursive solution. Evaluating (379-381) in order and then alternating between (382) and (383), we find for the first few terms

$$p_{1,1} = \alpha p_o, \tag{386}$$

$$p_{1,2} = (\alpha^2+\alpha)p_o, \tag{387}$$

$$p_{2,1} = (\alpha^3+2\alpha^2)p_o, \tag{388}$$

$$p_{2,2} = (\alpha^4+3\alpha^3+\alpha^2)p_o, \tag{389}$$

$$p_{3,1} = (\alpha^5+4\alpha^4+3\alpha^3)p_o, \tag{390}$$

$$p_{3,2} = (\alpha^6+5\alpha^5+6\alpha^4+\alpha^3)p_o, \tag{391}$$

$$p_{4,1} = (\alpha^7+6\alpha^6+10\alpha^5+4\alpha^4)p_o, \tag{392}$$

$$p_{4,2} = (\alpha^8+7\alpha^7+15\alpha^6+10\alpha^5+\alpha^4)p_o. \tag{393}$$

The values of $p_{4,j}$ in Equations (392-393) show that the general formula defining the $p_{n,j}$ is not of a trivial form as might first be expected from the pattern of Equations (386-391) considered alone. By trial and error, we find for Equations (386-393),

$$p_{n,j} = p_o \sum_{k=0}^{n+j-2} \frac{(n+k)!\alpha^{n+k}}{(2k-j+2)!(n+j-k-2)!}. \tag{394a}$$

Equation (394a) is in fact the solution to Equations (379-383) as can be verified by induction. Now, in order to determine the values of n and j for which Equation (394a) applies, we observe that the $p_{n,j}$ of Equations (386-393) are given in the order in which they *must* be determined when a recursive approach is used to express the $p_{n,j}$ in terms of the unknown p_o. Continuing the computations we would obtain from (382-383), $p_{5,1}$, $p_{5,2}$, $p_{6,1}$, $\cdots$, $p_{S-1,1}$, $p_{S-1,2}$, $p_{S,1}$, $p_{S,2}$, and finally $p_{S+1,1}$. Thus, since (379-383) are satisfied by (394a), it follows that the permissible subscript combinations in (394a) are

$$\left.\begin{array}{ll} n=1,2,3,\cdots,S+1 & \text{for } j=1; \\ n=1,2,3,\cdots,S & \text{for } j=2. \end{array}\right\} \tag{394b}$$

However, (394b) is just the range of subscript combinations which appear in all of Equations (382-383). Hence, we may regard Equations (382-383) as being general systems equations and Equations (379-381) as being the associated boundary conditions.

It is apparent that Equations (384) and (385) cannot both be boundary conditions since at least one of them must be used to compute the not yet determined probability $p_{S+1,2}$. Selecting (385) for this purpose, we obtain with the aid of (394),

$$p_{S+1,2} = \alpha p_{S,2} = p_o \sum_{k=0}^{S} \frac{(S+k)!\alpha^{S+k+1}}{(2k)!(S-k)!} \; . \tag{395}$$

Equation (384) remains to be satisfied. In a general solution, it might be a fourth boundary condition so that the ratios $p_{n,j}/p_o$ could be uniquely determined if that general solution possessed four unknown constants. In the present case, Equation (384) is redundant (as it must be for Equations (379-385) to be a consistent set). From (384) with the aid of (394), we have

$$p_{S+1,2} = p_{S+1,1} - \alpha p_{S,1}$$

$$= p_o \sum_{k=0}^{S} \frac{(S+k+1)!\alpha^{S+k+1}}{(2k+1)!(S-k)!} - \alpha p_o \sum_{k=0}^{S-1} \frac{(S+k)!\alpha^{S+k}}{(2k+1)!(S-k-1)!}$$

$$= \alpha p_o \left\{ \frac{(2S+1)!\alpha^{2S}}{(2S+1)!0!} + \sum_{k=0}^{S-1} \left[\frac{(S+k+1)!\alpha^{S+k}}{(2k+1)!(S-k)!} - \frac{(S+k)!\alpha^{S+k}}{(2k+1)!(S-k-1)!} \right] \right\}$$

$$= \alpha p_o \left\{ \alpha^{2S} + \sum_{k=0}^{S-1} \left[\frac{(S+k)!\alpha^{S+k}}{(2k)!(S-k)!} \left(\frac{S+k+1}{2k+1} - \frac{S-k}{(2k+1)} \right) \right] \right\}$$

$$= \alpha p_o \left\{ \frac{(2S)!\alpha^{2S}}{(2S)!0!} + \sum_{k=0}^{S-1} \frac{(S+k)!\alpha^{S+k}}{(2k)!(S-k)!} \right\}$$

$$= \alpha p_{S,2}, \tag{396}$$

as before.

The complete solution is thus given by Equations (394-395) when

p_o is evaluated from the normalizing condition

$$1 = p_o + \sum_{n=1}^{S+1} \sum_{j=1}^{2} p_{n,j}. \tag{397}$$

The unconditional state probabilities p_n may now be obtained by summing Equations (394-395) over j=1,2. In general, we have

$$p_n = p_{n,1} + p_{n,2}, \quad n=1,2,3,\cdots,S+1; \tag{398}$$

so that

$$p_n = p_o \sum_{k=0}^{n-1} \frac{(n+k)!\alpha^{n+k}}{(2k+1)!(n-k-1)!} + p_o \sum_{k=0}^{n} \frac{(n+k)!\alpha^{n+k}}{(2k)!(n-k)!}, \tag{399}$$

$$n=1,2,3,\cdots,S;$$

$$p_{S+1} = p_o \sum_{k=0}^{S} \frac{(S+k+1)!\alpha^{S+k+1}}{(2k+1)!(S-k)!} + \alpha p_o \sum_{k=0}^{S} \frac{(S+k)!\alpha^{S+k}}{(2k)!(S-k)!}. \tag{400}$$

Grouping terms, we can write Equations (399,400) in the forms

$$\left.\begin{aligned} p_n &= p_o \sum_{k=0}^{n} \frac{(n+k+1)!\alpha^{n+k}}{(2k+1)!(n-k)!}, \quad n=1,2,3,\cdots,S; \\ p_{S+1} &= p_o \sum_{k=0}^{S} \frac{(S+k+1)!(S+3k+2)\alpha^{S+k+1}}{(2k+1)!(S-k)!(S+k+1)}; \end{aligned}\right\} \tag{401}$$

where p_o is to be evaluated from the requirement,

$$1 = \sum_{n=0}^{S+1} p_n. \tag{402}$$

<u>A More Formal Approach</u>. From Equations (382), we obtain

$$-(\alpha+1)p_{n,2} = -(\alpha+1)^2 p_{n,1} + \alpha(\alpha+1)p_{n-1,1}, \tag{403}$$

$$\alpha p_{n-1,2} = \alpha(\alpha+1)p_{n-1,1} - \alpha^2 p_{n-2,1}. \tag{404}$$

Substituting these values into (383), we find

$$0 = p_{n+1,1} - (\alpha+1)^2 p_{n,1} + 2\alpha(\alpha+1)p_{n-1,1} - \alpha^2 p_{n-2,1}. \tag{405}$$

Equation (405) may be recognized as being a third-order, linear, homogeneous difference equation, in n, with constant coefficients. It admits to solutions of the form

$$p_{n,1} = C\beta^n, \tag{406}$$

where C is an arbitrary constant. Substituting (406) into (405) and eliminating common factors, we obtain the condition

$$0 = \beta^3 - (\alpha+1)^2\beta^2 + 2\alpha(\alpha+1)\beta - \alpha^2. \tag{407}$$

Now, we perform the same operations for $p_{n,2}$. From Equations (383), we obtain

$$-(\alpha+1)p_{n,1} = -(\alpha+1)^2 p_{n-1,2} + \alpha(\alpha+1)p_{n-2,2}, \tag{408}$$

$$\alpha p_{n-1,1} = \alpha(\alpha+1)p_{n-2,2} - \alpha^2 p_{n-3,2}. \tag{409}$$

Substituting these values into Equation (382), we find

$$0 = p_{n,2} - (\alpha+1)^2 p_{n-1,2} + 2\alpha(\alpha+1)p_{n-2,2} - \alpha^2 p_{n-3,2}. \tag{410}$$

However, this equation is of the same form as (405) so that, if we make the substitution

$$p_{n,2} = C\beta^n, \tag{411}$$

we again obtain Equation (407). Therefore, solutions to Equations (382-383) must be of the form

$$p_{n,j} = A_j\beta^n. \tag{412}$$

Substituting (412) into (382-383) and simplifying the resulting equations, we obtain

$$0 = A_2\beta - (\alpha+1)A_1\beta + \alpha A_1, \tag{413}$$

$$0 = A_1\beta^2 - (\alpha+1)\beta A_2 + \alpha A_2. \tag{414}$$

From the first of these, we find

$$A_2 = [(\alpha+1)\beta-\alpha]\beta^{-1} A_1. \tag{415}$$

Substitution of (415) into (414) will again produce Equation (407). Note that Equation (407) can be written in the form

$$0 = \beta - \{[(\alpha+1)\beta-\alpha]\beta^{-1}\}^2, \tag{416}$$

so that, if we set

$$\gamma \doteq [(\alpha+1)\beta-\alpha]\beta^{-1} \tag{417}$$

in Equations (415-416), we have

$$A_2 = \gamma A_1 \quad \text{and} \quad \beta = \gamma^2. \tag{418}$$

The relations (418) imply that the trial solution (412) can be replaced by

$$P_{n,j} = C\gamma^{2n+j-1}, \tag{419}$$

where the unknown factor C is independent of n and j.

We have γ defined in terms of β. However, the condition placed on values of γ is not nearly so complex as suggested by Equations

(416-418). Substituting (419) into either (382) or (383) and simplifying the result, we find

$$0 = \gamma^3 - (\alpha+1)\gamma^2 + \alpha. \tag{420}$$

Let the three roots of Equation (420) be denoted by γ_0, γ_1, and γ_2. Then, we have

$$\gamma_o = 1, \tag{421}$$

$$\gamma_1 = \frac{\alpha + \sqrt{\alpha^2+4\alpha}}{2}, \tag{422}$$

$$\gamma_2 = \frac{\alpha - \sqrt{\alpha^2+4\alpha}}{2}. \tag{423}$$

Therefore, the general solution to (382-383) is

$$p_{n,j} = C_o + C_1\gamma_1^{2n+j-1} + C_2\gamma_2^{2n+j-1} \tag{424}$$

$$n=1,2,3,\cdots,S+1 \quad \text{for } j=1,$$

$$\text{and} \quad n=1,2,3,\cdots,S \quad \text{for } j=2;$$

where C_o, C_1, and C_2 are arbitrary constants.

Substituting Equation (424) into the boundary Equations (379-381), we find

$$0 = C_o + C_1\gamma_1^2 + C_2\gamma_2^2 - \alpha p_o, \tag{425}$$

$$0 = C_o(-\alpha) + C_1[\gamma_1^3-(\alpha+1)\gamma_1^2] + C_2[\gamma_2^3-(\alpha+1)\gamma_2^2], \tag{426}$$

$$0 = C_o(-\alpha) + C_1\gamma_1[\gamma_1^3-(\alpha+1)\gamma_1^2] + C_2\gamma_2[\gamma_2^3-(\alpha+1)\gamma_2^2] + \alpha p_o. \tag{427}$$

However, in view of (420), these relations reduce to

$$\alpha p_o = C_o + C_1\gamma_1^2 + C_2\gamma_2^2, \tag{428}$$

$$0 = C_o + C_1 + C_2, \tag{429}$$

$$p_o = C_o + C_1\gamma_1 + C_2\gamma_2. \tag{430}$$

Solving Equations (428-430), we obtain

$$C_o = 0, \qquad C_1 = -C_2 = p_o(\alpha^2+4\alpha)^{-1/2}. \tag{431}$$

Since $C_o = 0$, the root $\gamma_o = 1$ is effectively eliminated from the solution. This result is highly desirable from the standpoint of our solution being consistent with that for the $M|E_2|1$ model. More specifically, as $S\to\infty$ and with the additional assumption that $2\alpha = (N\lambda/\mu)<1$ to ensure convergence,* our solution will become that for the $M|E_2|1$ (with $\lambda_n = N\lambda$).

Equation (431) can be used to verify the claim that one of

*The condition $2\alpha<1$ is not *mathematically* necessary to the present model since it possesses a finite number of states. However, the only practical cases will be found to be those for which the condition holds.

Equations (384,385) is redundant. To show this, we divide Equation (420) by $\gamma - 1$ to find the equation satisfied by γ_1 and γ_2, *viz.*,

$$0 = \gamma^2 - \alpha\gamma - \alpha = (\gamma-\gamma_1)(\gamma-\gamma_2). \tag{432}$$

Now, one of Equations (384,385) must be used to determine $p_{S+1,2}$. Eliminating $p_{S+1,2}$ from (384,385), we obtain the condition

$$0 = \alpha p_{S,2} - p_{S+1,1} + \alpha p_{S,1}, \tag{433}$$

in which all terms are known from Equations (421-424,431). We can directly verify that (433) is satisfied; however, it will be more illuminating to establish the permissible values of γ for which (433) is satisfied by the trial solution (419). Substituting (419) into (433), we find

$$0 = C\gamma^{2S}(\alpha\gamma-\gamma^2+\alpha), \tag{434}$$

which is seen to be equivalent to (432).

Grouping the results obtained, we have

$$\left.\begin{aligned} p_{n,j} &= p_o(\alpha^2+4\alpha)^{-1/2}\left\{\left[\frac{\alpha+\sqrt{\alpha^2+4\alpha}}{2}\right]^{2n+j-1} - \left[\frac{\alpha-\sqrt{\alpha^2+4\alpha}}{2}\right]^{2n+j-1}\right\}, \\ &\qquad n=1,2,3,\cdots,S+1 \quad \text{for } j=1, \\ &\qquad \text{and } n=1,2,3,\cdots,S \quad \text{for } j=2; \\ &\qquad \vdots \end{aligned}\right\} \tag{435}$$

$$p_{S+1,2} = \alpha p_{S,2} = p_o \alpha(\alpha^2+4\alpha)^{-1/2}\left\{\left[\frac{\alpha+\sqrt{\alpha^2+4\alpha}}{2}\right]^{2S+j-1} - \left[\frac{\alpha-\sqrt{\alpha^2+4\alpha}}{2}\right]^{2S+j-1}\right\}$$

where p_o is given by the normalizing condition (397). The unconditional state probabilities p_n may be obtained from (435) with the aid of (398). The equivalence of Equations (394-395) and (435) follows from a simple application of the binomial identity (309).

Solution for Arbitrary J

Form of the Solution. The first step in finding a solution to Equations (372-378) for arbitrary J will be to find a general solution to the system equations (375-376) using the approach illustrated for the case J = 2. Noting that (375-376) have constant coefficients and noting the interrelationship of subscripts, we suspect a solution of the form

$$P_{n,j} = A_j \beta^n. \tag{436}$$

Substituting (436) into (375-376), we find

$$0 = A_{j+1}\beta - (\alpha+1)A_j\beta + \alpha A_j, \tag{437}$$

$$0 = A_1\beta^2 - (\alpha+1)A_J\beta + \alpha A_J; \tag{438}$$

or

$$A_{j+1} = [\beta(\alpha+1)-\alpha]\beta^{-1}A_j, \quad j=1,2,3,\cdots,J-1; \tag{439}$$

$$A_1 = [\beta(\alpha+1)-\alpha]\beta^{-2}A_J. \tag{440}$$

Evaluating (439) recursively leads to

$$A_j = \left[\frac{\beta(\alpha+1)-\alpha}{\beta}\right]^{j-1} A_1, \quad j=2,3,4,\cdots,J; \tag{441}$$

where, in particular,

$$A_J = \left[\frac{\beta(\alpha+1)-\alpha}{\beta}\right]^{J-1} A_1. \tag{442}$$

Substitution of (442) into (440) and elimination of common factors yields the condition

$$0 = \beta - \left[\frac{\beta(\alpha+1)-\alpha}{\beta}\right]^{J}. \tag{443}$$

However, comparing (436,441,443), we see that a simplification is possible if we let

$$\gamma \doteq [\beta(\alpha+1)-\alpha]\beta^{-1}. \tag{444}$$

By (443) we have

$$\beta^n = \gamma^{nJ}, \tag{445}$$

and by (441) we have

$$A_j = \gamma^{j-1}A_1, \quad j=2,3,4,\cdots,J; \tag{446}$$

so that (436) can be replaced by

$$p_{n,j} = C\gamma^{Jn+j-1}, \tag{447}$$

where C is an unknown constant.

Substituting (447) into either of Equations (375,376) results in the condition,

$$\gamma^{J+1} - (\alpha+1)\gamma^J + \alpha = 0. \tag{448}$$

Roots of the Characteristic Equation. A number of observations may be made about the roots of Equation (448). The equation,

$$f(\gamma) \equiv \gamma^{J+1} - (\alpha+1)\gamma^J + \alpha = 0, \tag{449}$$

is a real algebraic equation. It has two pairwise changes of signs in its coefficients and, thus, by Descartes' Rule of Signs, it has either zero or two positive real roots. Clearly $\gamma = 1$ is a root; hence, the number of positive real roots must be two. Further,

$$f(-\gamma) = (-\gamma)^{J+1} - (\alpha+1)(-\gamma)^J + \alpha \tag{450}$$

has one change of sign if J is even and zero changes of sign if J is odd. Consequently, $f(\gamma)$ has one negative real root if J is even and none otherwise. Since f(x) is a *real* polynomial, the remaining roots are complex and occur in conjugate pairs.

Another observation is that all the roots except $\gamma = 1$ lie within the circle $|\gamma| = 1$ if $J\alpha<1$. To show this, we divide (448) by $\gamma - 1$ to obtain the equation satisfied by the remaining roots, *viz.*,

$$\gamma^J - \alpha(\gamma^{J-1}+\gamma^{J-2}+\gamma^{J-3}+\cdots+\gamma+1) = 0. \tag{451}$$

Suppose that $|\gamma| \geq 1$. Then, from (451) it follows that

$$\begin{aligned} |\gamma|^J &= \alpha|\gamma^{J-1} + \gamma^{J-2} + \gamma^{J-3} + \cdots + \gamma + 1| \\ &\leq \alpha\{|\gamma|^{J-1} + |\gamma|^{J-2} + |\gamma|^{J-3} + \cdots + |\gamma| + 1\} \\ &\leq J\alpha|\gamma|^{J-1}, \end{aligned} \tag{452}$$

which contradicts the assumption that $J\alpha<1$. Therefore, it must be the case that all roots of Equation (448), except for the root $\gamma = 1$, are less than unity in absolute value.

Finally, we observe that a sufficient condition for the roots of Equation (448) to be distinct is that $J\alpha<1$. Under the (one-to-one) transformation

$$\gamma = (1+\alpha)(1-z), \tag{453}$$

Equation (448) becomes

$$(1-z)^J z = (1-z_o)^J z_o, \tag{454}$$

where z_o is the value of z corresponding to $\gamma = 1$, namely,

$$z_o = \alpha/(1+\alpha). \tag{455}$$

Equations (454-455), with $J\alpha<1$, are a special case of the equation

$$(1-z)^m z^n = (1-z_o)^m z_o^n, \tag{456}$$

which was studied in some detail by Syski ([212], pp.313-317) who showed that (456) has precisely m roots with real parts greater than z_o, that these m roots are distinct, and that the m roots lie within the circle $|1-z| = 1 - z_o$. In the present problem, Equations (453,455) show that the m = J roots must be bounded by the condition

$$(1+\alpha)^{-1} = 1 - z_o > |1-z| = |\gamma|(1+\alpha)^{-1}, \tag{457}$$

or

$$|\gamma| < 1. \tag{458}$$

As we saw previously, this is indeed the case when $J\alpha<1$. It follows that Equation (448) has J + 1 distinct roots (including the root $\gamma = 1$) when $J\alpha<1$.

<u>The General Solution</u>. We assume that $J\alpha<1$ so that the J + 1

roots of Equation (448) are distinct. One root is $\gamma = 1$. Let the remaining roots be denoted by $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_J$. Then, the general solution to Equations (375-376) can be expressed as

$$p_{n,j} = \left[C_o + \sum_{r=1}^{J} C_r \gamma_r^{Jn+j-1}\right] p_o, \tag{459}$$

$$n=1,2,3,\cdots,S+1 \quad \text{for } j=1,$$

$$\text{and} \quad n=1,2,3,\cdots,S \quad \text{for } j=2,3,4,\cdots,J;$$

where $C_o, C_1, C_2, \cdots, C_J$ are arbitrary constants.

<u>Elimination of C_o.</u> As a first step in evaluating the unknown constants, we show that $C_o = 0$. Consider Equations (377-378). There are J equations of which J - 1 must be used to determine the $p_{S+1,j}$ for $j=2,3,4,\cdots,J$. It follows that we can eliminate these $p_{S+1,j}$ from one of the equations to determine a boundary condition on the general solution (459). Summing (377) over $j=1,2,3,\cdots,J-1$, we obtain

$$0 = \sum_{j=1}^{J-1} (p_{S+1,j+1} - p_{S+1,j} + \alpha p_{S,j})$$

$$= p_{S+1,J} - p_{S+1,1} + \alpha \sum_{j=1}^{J-1} p_{S,j}. \tag{460}$$

From (378), we have $p_{S+1,J} = \alpha p_{S,J}$ so that (460) becomes

$$0 = p_{S+1,1} - \alpha \sum_{j=1}^{J} p_{S,j}. \tag{461}$$

Substituting the appropriate values from (459) into (461) produces

$$0 = \left[C_o + \sum_{r=1}^{J} C_r \gamma_r^{JS+J}\right] - \alpha \sum_{j=1}^{J} \left[C_o + \sum_{r=1}^{J} C_r \gamma_r^{JS+j-1}\right]$$

$$= C_o(1-J\alpha) + \sum_{r=1}^{J} \left\{ C_r \gamma_r^{JS} \left[\gamma_r^{J} - \alpha \sum_{j=1}^{J} \gamma_r^{j-1}\right]\right\} . \qquad (462)$$

However, in view of the fact that (451) is satisfied by each of the γ_r $(r=1,2,3,\cdots,J)$, the innermost bracketed expression in (462) vanishes for each r $(r=1,2,3,\cdots,J)$. Consequently, Equation (462) reduces to

$$0 = C_o(1+J\alpha), \quad \text{or} \quad C_o = 0 \qquad (463)$$

since $J\alpha<1$. Therefore, the general solution (459) simplifies to

$$P_{n,j} = P_o \sum_{r=1}^{J} C_r \gamma_r^{Jn+j-1}, \qquad (464)$$

$n=1,2,3,\cdots,S+1$ for $j=1$,

and $n=1,2,3,\cdots,S$ for $j=2,3,4,\cdots,J$.

<u>Determination of the C_r</u>. It remains to satisfy the J + 1 boundary conditions (372-374). Since there are only J unknowns $C_1, C_2, C_3, \cdots, C_J$, it is apparent that one of these boundary conditions must be redundant in order for Equations (372-378) to be a consistent set.

Indeed, this is the case. Substituting (464) into (372), we find

$$\alpha p_o = p_{1,1} = p_o \sum_{r=1}^{J} C_r \gamma_r^J. \tag{465}$$

However, this result can be deduced independently from Equations (373-374): Note that by Equation (448) we have

$$-\alpha = \gamma_r^{J+1} - (\alpha+1)\gamma_r^J, \qquad r=1,2,3,\cdots,J. \tag{466}$$

Thus, substituting (464) into (373), we obtain

$$\begin{aligned} 0 &= p_{1,j+1} - (\alpha+1)p_{1,j} \\ &= p_o \sum_{r=1}^{J} C_r\left[\gamma_r^{J+j} - (\alpha+1)\gamma_r^{J+j-1}\right] \\ &= p_o \sum_{r=1}^{J} C_r \gamma_r^{j-1}\left[\gamma_r^{J+1} - (\alpha+1)\gamma_r^{J}\right] \\ &= \alpha p_o \sum_{r=1}^{J} C_r \gamma_r^{j-1}, \qquad j=1,2,3,\cdots,J-1; \end{aligned} \tag{467}$$

and similarly, from (374), we obtain

$$\begin{aligned} -\alpha p_o &= p_{2,1} - (\alpha+1)p_{1,J} \\ &= p_o \sum_{r=1}^{J} C_r\left[\gamma_r^{2J} - (\alpha+1)\gamma_r^{2J-1}\right] \end{aligned}$$

$$= p_o \sum_{r=1}^{J} C_r \gamma_r^{J-1} \left[\gamma_r^{J+1} - (\alpha+1)\gamma_r^{J} \right]$$

$$= -\alpha p_o \sum_{r=1}^{J} C_r \gamma_r^{J-1}. \tag{468}$$

However, the sum of Equations (467) minus Equation (468) is

$$\alpha p_o = \alpha p_o \sum_{r=1}^{J} C_r \gamma_r^{J-1} + \sum_{j=1}^{J-1} \left[\alpha p_o \sum_{r=1}^{J} C_r \gamma_r^{j-1} \right]$$

$$= p_o \sum_{r=1}^{J} \left[C_r \alpha \sum_{j=1}^{J} \gamma_r^{j-1} \right]$$

$$= p_o \sum_{r=1}^{J} C_r \gamma_r^{J} \tag{469}$$

in view of Equation (451). The identity of Equations (465) and (469) establishes the claimed redundancy of one of Equations (372-374).

Equations (467-468) will be used to determine the C_r. The required relationships can be illustrated more graphically by the matrix equation,

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_J \\ \gamma_1^2 & \gamma_2^2 & \gamma_3^2 & \cdots & \gamma_J^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_1^{J-2} & \gamma_2^{J-2} & \gamma_3^{J-2} & \cdots & \gamma_J^{J-2} \\ \gamma_1^{J-1} & \gamma_2^{J-1} & \gamma_3^{J-1} & \cdots & \gamma_J^{J-1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_{J-1} \\ C_J \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \qquad (470)$$

The matrix of coefficients in (470) is of the Vandermonde type and accordingly has the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_J \\ \gamma_1^2 & \gamma_2^2 & \gamma_3^2 & \cdots & \gamma_J^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_1^{J-2} & \gamma_2^{J-2} & \gamma_3^{J-2} & \cdots & \gamma_J^{J-2} \\ \gamma_1^{J-1} & \gamma_2^{J-1} & \gamma_3^{J-1} & \cdots & \gamma_J^{J-1} \end{vmatrix} = \prod_{1 \le k < m \le J} (\gamma_m - \gamma_k). \qquad (471)$$

By Cramer's Rule, the value of C_r is

$$C_r = N_r/D, \quad r=1,2,3,\cdots,J; \qquad (472)$$

where N_r is the determinant

$$N_r = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & 0 & \gamma_{r+1} & \gamma_{r+2} & \cdots & \gamma_J \\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_{r-1}^2 & 0 & \gamma_{r+1}^2 & \gamma_{r+2}^2 & \cdots & \gamma_J^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_1^{J-2} & \gamma_2^{J-2} & \cdots & \gamma_{r-1}^{J-2} & 0 & \gamma_{r+1}^{J-2} & \gamma_{r+2}^{J-2} & \cdots & \gamma_J^{J-2} \\ \gamma_1^{J-1} & \gamma_2^{J-1} & \cdots & \gamma_{r-1}^{J-1} & 1 & \gamma_{r+1}^{J-1} & \gamma_{r+2}^{J-1} & \cdots & \gamma_J^{J-1} \end{vmatrix} . \quad (473)$$

However, the determinant (473) can be expanded in terms of minors of the *rth* column to obtain

$$N_r = (-1)^{J+r} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & \gamma_{r+1} & \gamma_{r+2} & \cdots & \gamma_J \\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_{r-1}^2 & \gamma_{r+1}^2 & \gamma_{r+2}^2 & \cdots & \gamma_J^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_1^{J-2} & \gamma_2^{J-2} & \cdots & \gamma_{r-1}^{J-2} & \gamma_{r+1}^{J-2} & \gamma_{r+2}^{J-2} & \cdots & \gamma_J^{J-2} \end{vmatrix}$$

$$= (-1)^{J+r} \prod_{\substack{1 \le k < m \le J \\ k \ne r \\ m \ne r}} (\gamma_m - \gamma_k), \quad (474)$$

in view of (471).

Substituting (471,474) into (472), we find

$$C_r = \frac{(-1)^{J+r} \prod\limits_{\substack{1\le k<m\le J \\ k\ne r, m\ne r}} (\gamma_m-\gamma_k)}{\prod\limits_{1\le k<m\le J} (\gamma_m-\gamma_k)}$$

$$= (-1)^{J+r}[(\gamma_J-\gamma_r)(\gamma_{J-1}-\gamma_r)\cdots(\gamma_{r+1}-\gamma_r)(\gamma_r-\gamma_{r-1})(\gamma_r-\gamma_{r-2})\cdots(\gamma_r-\gamma_1)]^{-1}$$

$$= (-1)^{2r}[(\gamma_r-\gamma_J)(\gamma_r-\gamma_{J-1})\cdots(\gamma_r-\gamma_{r+1})(\gamma_r-\gamma_{r-1})(\gamma_r-\gamma_{r-2})\cdots(\gamma_r-\gamma_1)]^{-1}$$

$$= \prod_{\substack{1\le k\le J \\ k\ne r}} (\gamma_r-\gamma_k)^{-1}. \tag{475}$$

The expression (475) for the C_r can be simplified somewhat. Note that Equation (449) may also be written in the factored form

$$f(\gamma) = \prod_{k=0}^{J} (\gamma-\gamma_k) = 0, \tag{476}$$

where $\gamma_o \doteq 1$. Then,

$$f'(\gamma_r) \doteq \left.\frac{df(\gamma)}{d\gamma}\right|_{\gamma=\gamma_r} = \prod_{\substack{0\le k\le J \\ k\ne r}} (\gamma_r-\gamma_k) = (\gamma_r-1)/C_r, \quad r=1,2,3,\cdots,J. \tag{477}$$

But, from (449), we have also

$$f'(\gamma_r) = (J+1)\gamma_r^J - J(\alpha+1)\gamma_r^{J-1} = \gamma_r^{J-1}[(J+1)\gamma_r-J(\alpha+1)], \tag{478}$$

$$r=1,2,3,\cdots,J.$$

Hence, comparing (477) with (478), we find

$$C_r = (1-\gamma_r)\gamma_r^{1-J}[J(\alpha+1)-(J+1)\gamma_r]^{-1}, \quad r=1,2,3,\cdots,J. \tag{479}$$

Substituting these values into Equation (464), we obtain

$$p_{n,j} = p_o \sum_{r=1}^{J}\left\{(1-\gamma_r)\gamma_r^{J(n-1)+j}[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}, \tag{480}$$

$$n=1,2,3,\cdots,S+1 \quad \text{for } j=1,$$

$$n=1,2,3,\cdots,S \quad \text{for } j=2,3,4,\cdots,J,$$

under the condition that $J\alpha<1$.

<u>Determination of the Remaining $P_{n,j}$</u>. Replacing j with k in Equation (377) and summing the result over $k=j,j+1,j+2,\cdots,J-1$ leads to

$$0 = \sum_{k=j}^{J-1} (p_{S+1,k+1}-p_{S+1,k}+\alpha p_{S,k})$$

$$= p_{S+1,J} - p_{S+1,j} + \alpha \sum_{k=j}^{J-1} p_{S,k}, \quad j=1,2,3,\cdots,J-1. \tag{481}$$

Combining (378) and (481) produces

$$p_{S+1,j} = \alpha \sum_{k=j}^{J} p_{S,k}, \quad j=1,2,3,\cdots,J; \tag{482}$$

where the $p_{S,k}$ are known from Equation (480).

Therefore,

$$p_{S+1,j} = \alpha p_o \sum_{k=j}^{J} \sum_{r=1}^{J} \left\{(1-\gamma_r)\gamma_r^{JS-J+k}[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}$$

$$= \alpha p_o \sum_{r=1}^{J} \left\{(1-\gamma_r)[J(\alpha+1)-(J+1)\gamma_r]^{-1} \sum_{k=j}^{J} \gamma_r^{JS-J+k}\right\}$$

$$= \alpha p_o \sum_{r=1}^{J} \left\{(\gamma_r^{JS-J+j}-\gamma_r^{JS+1})[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}, \qquad (483)$$

$$j=1,2,3,\cdots,J.$$

Results Collected. We now have the complete solution in Equations (480,483). For $J\alpha = (N\lambda/\mu) < 1$,

$$\left.\begin{array}{l} p_{n,j} = p_o \displaystyle\sum_{r=1}^{J} \left\{(1-\gamma_r)\gamma_r^{J(n-1)+j}[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}, \\ \qquad\qquad n=1,2,3,\cdots,S \text{ and } j=1,2,3,\cdots,J; \\ p_{S+1,j} = \alpha p_o \displaystyle\sum_{r=1}^{J} \left\{(\gamma_r^{JS-J+j}-\gamma_r^{JS+1})[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}, \\ \qquad\qquad j=1,2,3,\cdots,J; \end{array}\right\} \qquad (484)$$

where the γ_r are the roots of Equation (451) and where p_o is to be evaluated from the normalizing condition

$$1 = p_o + \sum_{n=1}^{S+1} \sum_{j=1}^{J} p_{n,j}. \qquad (485)$$

The Unconditional State Probabilities. The probabilities p_n, of there being n failed machines in the system irrespective of the service phase, are obtained by summing the $p_{n,j}$ over all possible phases $j=1,2,3,\cdots,J$. Thus, with the aid of (448) in simplifying the expression for p_{S+1}, we have from (484),

$$\left.\begin{aligned} p_n &= p_o \sum_{r=1}^{J} \left\{\gamma_r^{J(n-1)+1}(1-\gamma_r^J)[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}, \\ &\qquad\qquad n=1,2,3,\cdots,S; \\ p_{S+1} &= p_o(1-J\alpha) \sum_{r=1}^{J}\left\{\gamma_r^{JS+1}[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}; \end{aligned}\right\} \tag{486}$$

where $J\alpha = (N\lambda/\mu)<1$, where the γ_r are the roots of Equation (451), and where p_o is to be evaluated from the normalizing condition

$$\sum_{n=0}^{S+1} p_n = 1. \tag{487}$$

In the present problem, the terms of the sum (487) form geometric progressions in $(\gamma_r^J)^n$ so that we have simply

$$p_o = \left\{1 + \sum_{r=1}^{J}\left[\frac{\gamma_r\left(1-J\alpha\gamma_r^{JS}\right)}{J(\alpha+1)-(J+1)\gamma_r}\right]\right\}^{-1}. \tag{488}$$

When J = 1, the solution (486,488) specializes to the form (41-42) with c = 1 (Taylor and Jackson's single-server model [232]); and, when J = 2, to the form (401-402).

The Type II Model with Chi-Square Servicing

Description of the Model

In this section, a model will be derived for the Type II repairman system with spares in which: (a) the machine failure times follow Equation (2), a negative exponential distribution with mean $1/\lambda$; (b) the machine service times follow Equation (342), a scale-modified chi-square distribution with 2J degrees of freedom and mean $1/\mu$; and (c) there are N working positions for machines, N + S machines total, and a single-server. In essence, the situation of interest here is the same as in the preceding section except that the system incorporates the Case II operating policy under which, after all spares have been exhausted, a number of machines less than N continues to operate.

We will retain the notation and conventions of the preceding section. The λ_n and μ_n are the same as for Toft and Boothroyd's single-server model [240], that is, as given by Equations (57) with c = 1. We again apply the "method of successive stages" to artificially construct an elementary Markov process involving J(N+S) + 1 states. The $\lambda_{n,j}$ and $\mu_{n,j}$ are

$$\left.\begin{array}{lll} \lambda_o = N\lambda, & \mu_o = 0, & n=0; \\ \lambda_{n,j} = N\lambda, & \mu_{n,j} = J\mu, & n=1,2,3,\cdots,S \text{ and } j=1,2,3,\cdots,J; \\ \lambda_{n,j} = (N+S-n)\lambda, & \mu_{n,j} = J\mu, & n=S,S+1,S+2,\cdots,N+S \text{ and } j=1,2,3,\cdots,J. \end{array}\right\} \quad (489)$$

Formulation of Equations

We use the usual approach for Poisson queues in formulating the equations of state. Attention focuses on the possible happenings in a small interval of time Δt. If the phases are numbered in reverse order of their execution so that a unit enters service at phase J and completes service at phase 1, the state probabilities at times $t + \Delta t$ and t are related by

$$P_o(t+\Delta t) = (1-N\lambda\Delta t)P_o(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{1,1}(t) + \mathcal{O}(\Delta t); \tag{490}$$

$$P_{1,j}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{1,j}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{1,j+1}(t) + \mathcal{O}(\Delta t),\quad j=1,2,3,\cdots,J-1; \tag{491}$$

$$P_{1,J}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{1,J}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{2,1}(t) + (N\lambda\Delta t)P_o(t) + \mathcal{O}(\Delta t); \tag{492}$$

$$P_{n,j}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{n,j}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{n,j+1}(t) + (N\lambda\Delta t)(1-J\mu\Delta t)P_{n-1,j}(t) + \mathcal{O}(\Delta t), \tag{493}$$

$$n=2,3,4,\cdots,S \quad\text{and}\quad j=1,2,3,\cdots,J-1;$$

$$P_{n,J}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{n,J}(t) + (1-N\lambda\Delta t)(J\mu\Delta t)P_{n+1,1}(t) + (N\lambda\Delta t)(1-J\mu\Delta t)P_{n-1,J}(t) + \mathcal{O}(\Delta t), \tag{494}$$

$$n=2,3,4,\cdots,S-1;$$

$$P_{S,J}(t+\Delta t) = (1-N\lambda\Delta t)(1-J\mu\Delta t)P_{S,J}(t) + [1-(N-1)\lambda\Delta t](J\mu\Delta t)P_{S+1,1}(t)$$

$$+ (N\lambda\Delta t)(1-J\mu\Delta t)P_{S-1,J} + O(\Delta t); \qquad (495)$$

$$P_{n,j}(t+\Delta t) = [1-(N+S-n)\lambda\Delta t](1-J\mu\Delta t)P_{n,j}(t) + [1-(N+S-n)\lambda\Delta t]$$

$$\cdot(J\mu\Delta t)P_{n,j+1}(t) + [(N+S-n+1)\lambda\Delta t]$$

$$\cdot(1-J\mu\Delta t)P_{n-1,j}(t) + O(\Delta t), \qquad (496)$$

$$n=S+1,S+2,S+3,\cdots,N+S-1 \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$P_{n,J}(t+\Delta t) = [1-(N+S-n)\lambda\Delta t](1-J\mu\Delta t)P_{n,J}(t) + [1-(N+S-n-1)\lambda\Delta t]$$

$$\cdot(J\mu\Delta t)P_{n+1,1}(t) + [(N+S-n+1)\lambda\Delta t]$$

$$\cdot(1-J\mu\Delta t)P_{n-1,J}(t) + O(\Delta t), \qquad (497)$$

$$n=S+1,S+2,S+3,\cdots,N+S-1;$$

$$P_{N+S,j}(t+\Delta t) = (1-J\mu\Delta t)P_{N+S,j}(t) + (J\mu\Delta t)P_{N+S,j+1}(t)$$

$$+ (\lambda\Delta t)(1-J\mu\Delta t)P_{N+S-1,j}(t) + O(\Delta t), \qquad (498)$$

$$j=1,2,3,\cdots,J-1;$$

$$P_{N+S,J}(t+\Delta t) = (1-J\mu\Delta t)P_{N+S,J}(t) + (\lambda\Delta t)(1-J\mu\Delta t)P_{N+S-1,J}(t) + O(\Delta t); \qquad (499)$$

where $\mathcal{O}(\Delta t)$ denotes the inclusion of terms of a smaller order of magnitude than Δt.

Rearranging Equations (490-499) in the usual way to take advantage of relation (356) and taking the limit as $\Delta t \to 0$, we obtain the dynamic equations of the system,

$$\frac{dP_o(t)}{dt} = -N\lambda P_o(t) + J\mu P_{1,1}(t); \tag{500}$$

$$\frac{dP_{1,j}(t)}{dt} = -(N\lambda+J\mu)P_{1,j}(t) + J\mu P_{1,j+1}(t), \quad j=1,2,3,\cdots,J-1; \tag{501}$$

$$\frac{dP_{1,J}(t)}{dt} = -(N\lambda+J\mu)P_{1,J}(t) + J\mu P_{2,1}(t) + N\lambda P_o(t); \tag{502}$$

$$\frac{dP_{n,j}(t)}{dt} = -(N\lambda+J\mu)P_{n,j}(t) + J\mu P_{n,j+1}(t) + N\lambda P_{n-1,j}(t), \tag{503}$$

$$n=2,3,4,\cdots,S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$\frac{dP_{n,J}(t)}{dt} = -(N\lambda+J\mu)P_{n,J}(t) + J\mu P_{n+1,1}(t) + N\lambda P_{n-1,J}(t), \tag{504}$$

$$n=2,3,4,\cdots,S;$$

$$\frac{dP_{n,j}(t)}{dt} = -[(N+S-n)\lambda+J\mu]P_{n,j}(t) + J\mu P_{n,j+1}(t)$$

$$+ (N+S-n+1)\lambda P_{n-1,j}(t), \tag{505}$$

$$n=S+1,S+2,S+3,\cdots,N+S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$\frac{dP_{n,J}(t)}{dt} = -[(N+S-n)\lambda+J\mu]P_{n,J}(t) + J\mu P_{n+1,1}(t)$$

$$+ (N+S-n+1)\lambda P_{n-1,J}(t), \quad n=S+1,S+2,S+3,\cdots,N+S-1; \tag{506}$$

$$\frac{dP_{N+S,J}(t)}{dt} = -J\mu P_{N+S,J}(t) + \lambda P_{N+S-1,J}(t). \tag{507}$$

In the steady-state, Equations (500-507) become

$$0 = -N\lambda p_o + J\mu p_{1,1}; \tag{508}$$

$$0 = -(N\lambda+J\mu)p_{1,j} + J\mu p_{1,j+1}, \quad j=1,2,3,\cdots,J-1; \tag{509}$$

$$0 = -(N\lambda+J\mu)p_{1,J} + J\mu p_{2,1} + N\lambda p_o; \tag{510}$$

$$0 = -(N\lambda+J\mu)p_{n,j} + J\mu p_{n,j+1} + N\lambda p_{n-1,j}, \tag{511}$$

$$n=2,3,4,\cdots,S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$0 =-(N\lambda+J\mu)p_{n,J} + J\mu p_{n+1,1} + N\lambda p_{n-1,J}, \quad n=2,3,4,\cdots,S; \tag{512}$$

$$0 =-[(N+S-n)\lambda+J\mu]p_{n,j} + J\mu p_{n,j+1} + (N+S-n+1)\lambda p_{n-1,j}, \tag{513}$$

$$n=S+1,S+2,S+3,\cdots,N+S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$0 = -[(N+S-n)\lambda+J\mu]p_{n,J} + J\mu p_{n+1,1} + (N+S-n+1)\lambda p_{n-1,J}, \qquad (514)$$

$$n=S+1,S+2,S+3,\cdots,N+S-1;$$

$$0 = -J\mu p_{N+S,J} + \lambda p_{N+S-1,J} . \qquad (515)$$

Making the substitutions,

$$\omega \doteq \frac{\lambda}{J\mu} \text{ and } \alpha \doteq N\omega = \frac{N\lambda}{J\mu} , \qquad (516)$$

and rearranging terms, we can express Equations (508-515) in the more convenient forms

$$0 = p_{1,1} - \alpha p_o; \qquad (517)$$

$$0 = p_{1,j+1} - (\alpha+1)p_{1,j}, \quad j=1,2,3,\cdots,J-1; \qquad (518)$$

$$0 = p_{2,1} - (\alpha+1)p_{1,J} + \alpha p_o; \qquad (519)$$

$$0 = p_{n,j+1} - (\alpha+1)p_{n,j} + \alpha p_{n-1,j}, \qquad (520)$$

$$n=2,3,4,\cdots,S \text{ and } j=1,2,3,\cdots,J-1;$$

$$0 = p_{n+1,1} - (\alpha+1)p_{n,J} + \alpha p_{n-1,J}, \quad n=2,3,4,\cdots,S; \qquad (521)$$

$$0 = p_{n,j+1} - [(N+S-n)\omega+1]p_{n,j} + (N+S-n+1)\omega p_{n-1,j}, \qquad (522)$$

$$n=S+1,S+2,S+3,\cdots,N+S \quad \text{and} \quad j=1,2,3,\cdots,J-1;$$

$$0 = p_{n+1,1} - [(N+S-n)\omega+1]p_{n,J} + (N+S-n+1)\omega p_{n-1,J}, \tag{523}$$

$$n=S+1,S+2,S+3,\cdots,N+S-1;$$

$$0 = p_{N+S,J} - \omega p_{N+S-1,J}. \tag{524}$$

Stationary Solution

Earlier Results Adapted. Observe that Equations (517-521) are identical in every respect to Equations (372-376). It follows that the *ratios* $p_{n,j}/p_o$ for Equations (517-521) must be the same as those obtained for Equations (372-376) in the previous section. Accordingly, we have from Equations (480), the relations

$$\frac{p_{n,j}}{p_o} = \sum_{r=1}^{J} \left\{(1-\gamma_r)\gamma_r^{J(n-1)+j}[J(\alpha+1)-(J+1)\gamma_r]^{-1}\right\}, \tag{525}$$

$$n=1,2,3,\cdots,S+1 \quad \text{for } j=1,$$

$$\text{and } n=1,2,3,\cdots,S \quad \text{for } j=2,3,4,\cdots,J;$$

where $J\alpha = (N\lambda/\mu)<1$, and where the γ_r $(r=1,2,3,\cdots,J)$ are the roots of the real polynomial (451).

Solution Rationale. The task is now reduced to obtaining the $p_{n,j}/p_o$ ratios for Equations (522-524). The most direct approach would consist of an unembellished recursive evaluation of the $p_{n,j}$; however, it can be seen that the required labor would be considerably reduced if general solutions could be developed for arbitrary j and fixed n. From (522) we have

$$p_{S+1,j+1} - [(N-1)\omega+1]p_{S+1,j} = -N\omega p_{S,j}, \tag{526}$$

$$p_{S+2,j+1} - [(N-2)\omega+1]p_{S+2,j} = -(N-1)\omega p_{S+1,j}, \tag{527}$$

$$p_{S+3,j+1} - [(N-3)\omega+1]p_{S+3,j} = -(N-2)\omega p_{S+2,j}, \tag{528}$$

$$p_{S+4,j+1} - [(N-4)\omega+1]p_{S+4,j} = -(N-3)\omega p_{S+3,j}, \tag{529}$$

etc. The $p_{S,j}$ are known from Equation (525). Suppose we were to solve Equation (526) for $p_{S+1,j}$, substitute this expression into Equation (527) and solve for $p_{S+2,j}$, and so on until finally $p_{N+S,j}$ was determined. At each step, we would be solving a linear, nonhomogeneous, first-order difference equation of the form

$$q_{j+1} - bq_j = \xi_j, \quad j=1,2,3,\cdots,J-1; \tag{530}$$

where ξ_j is a known function of j and b is independent of j.

The homogeneous solution $q_j^{(H)}$ must satisfy Equation (530) with ξ_j set to zero. Thus,

$$q_j^{(H)} = Gb^{j-1}, \quad j=1,2,3,\cdots,J; \tag{531}$$

where G is an arbitrary constant. A particular solution $q_j^{(P)}$ to Equation (530) may be obtained, by the method of variation of parameters (see Appendix), in the form

$$q_j^{(P)} = \sum_{k=1}^{j} b^{j-k}\xi_k, \quad j=1,2,3,\cdots,J. \tag{532}$$

Therefore, the total solution to Equation (530) may be expressed as

$$q_j = q_j^{(H)} + q_j^{(P)} = Gb^{j-1} + \sum_{k=1}^{j} b^{j-k}\xi_k, \quad j=1,2,3,\cdots,J. \tag{533}$$

It follows that general solutions to Equations (522) for $n=S+1,S+2,S+3,\cdots,N+S$ may be obtained from a recursive application of the relations

$$p_{n,j} = p_{n,j}^{(H)} + p_{n,j}^{(P)}, \qquad j=1,2,3,\cdots,J; \tag{534}$$

$$p_{n,j}^{(H)} = G_n[(N+S-n)\omega+1]^{j-1}, \qquad j=1,2,3,\cdots,J; \tag{535}$$

$$p_{n,j}^{(P)} = -(N+S-n+1)\omega \sum_{k=1}^{j} \{[(N+S-n)\omega+1]^{j-k} p_{n-1,k}\}, \tag{536}$$

$$j=1,2,3,\cdots,J;$$

where the G_n are arbitrary factors independent of j. One of the boundary conditions (523,524) can be used to determine the value of G_n at each step.

We will now use the above scheme to obtain explicit expressions for the $p_{n,j}$. As a first step, we will develop the pertinent relationships for n=S,S+1,S+2,and S+3. Define

$$K_r \doteq (1-\gamma_r)\gamma_r^{J(S-1)}[J(\alpha+1)-(J+1)\gamma_r]^{-1}, \quad r=1,2,3,\cdots,J. \tag{537}$$

Then, from (525), we establish that

$$p_{S,j} = p_o \sum_{r=1}^{J} K_r\gamma_r^j, \quad j=1,2,3,\cdots,J; \tag{538}$$

$$p_{S+1,1} = p_o \sum_{r=1}^{J} K_r\gamma_r^{J+1}. \tag{539}$$

Finally, for reasons pertinent to the method to be used below in obtaining particular solutions, we point out that, in view of the result (458), $[(N+S-n)\omega+1]$ is *not* a root of Equation (451) for any value of $n \le N+S$.

<u>Solution for n=S+1</u>. From Equations (526,538), we have

$$p_{S+1,j+1} - [(N-1)\omega+1]p_{S+1,j} = -N\omega p_{S,j} = -N\omega p_o \sum_{r=1}^{J} K_r\gamma_r^j, \tag{540}$$

$$j=1,2,3,\cdots,J-1;$$

which may be regarded as being a linear difference equation in j alone. The homogeneous solution $p_{S+1,j}^{(H)}$ must satisfy

$$p_{S+1,j+1}^{(H)} - [(N-1)\omega+1]p_{S+1,j}^{(H)} = 0, \quad j=1,2,3,\cdots,J-1. \tag{541}$$

Equations (541) may be evaluated recursively to yield

$$p_{S+1,j+1}^{(H)} = [(N-1)\omega+1]^j p_{S+1,1}^{(H)}, \quad j=1,2,3,\cdots,J-1. \tag{542}$$

But $p_{S+1,1}^{(H)}$ is an unknown constant, so we can also write

$$p_{S+1,j}^{(H)} = p_o G_{S+1}[(N-1)\omega+1]^{j-1}, \quad j=1,2,3,\cdots,J; \tag{543}$$

where G_{S+1} is independent of j.

To determine a particular solution of Equation (540), we write

$$p_{S+1,j}^{(P)} = p_o \sum_{r=1}^{J} A_{S+1,r}\gamma_r^j, \tag{544}$$

where the $A_{S+1,r}$ are independent of j, and apply the "method of undetermined coefficients."* Specifically, we substitute (544) into (540) to obtain

$$p_o \sum_{r=1}^{J} (A_{S+1,r}\gamma_r^{j+1}) - [(N-1)\omega+1]p_o \sum_{r=1}^{J} (A_{S+1,r}\gamma_r^j)$$

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*In the present case, particular solutions follow more readily from the method of undetermined coefficients than from the method of variation of parameters suggested earlier (see Equation (532)). The method of undetermined coefficients is applicable to linear nonhomogeneous difference equations having constant coefficients when the right-hand side is a linear combination of terms of the forms

$$a^j,\ e^{jb},\ \sin(cj),\ \cos(cj),\ j^m \quad (m=0,1,2,\cdots),$$

or of products of such terms. See, e.g., Hildebrand [83], pp.242-244.

$$= -N\omega p_o \sum_{r=1}^{J} K_r\gamma_r^j, \tag{545}$$

and equate coefficients of γ_r^j to establish that

$$A_{S+1,r} = N\omega K_r[(N-1)\omega+1-\gamma_r]^{-1}, \quad r=1,2,3,\cdots,J. \tag{546}$$

The total solution is thus

$$p_{S+1,j} = p_o G_{S+1}[(N-1)\omega+1]^{j-1} + p_o \sum_{r=1}^{J} A_{S+1,r}\gamma_r^j, \tag{547}$$

$$j=1,2,3,\cdots,J;$$

where only p_o and G_{S+1} remain to be determined.

In particular,

$$p_{S+1,1} = p_o G_{S+1} + p_o \sum_{r=1}^{J} A_{S+1,r}\gamma_r, \tag{548}$$

so that we have from (539,546,548),

$$p_o \sum_{r=1}^{J} K_r\gamma_r^{J+1} = p_o G_{S+1} + p_o \sum_{r=1}^{J} N\omega K_r\gamma_r[(N-1)\omega+1-\gamma_r]^{-1}, \tag{549}$$

from which it follows that

$$G_{S+1} = \sum_{r=1}^{J} \{K_r\gamma_r^{J+1} - N\omega K_r\gamma_r[(N-1)\omega+1-\gamma_r]^{-1}\}$$

$$= \sum_{r=1}^{J} \frac{K_r\gamma_r\{\gamma_r^J[(N-1)\omega+1-\gamma_r]-N\omega\}}{[(N-1)\omega+1-\gamma_r]}$$

$$= \sum_{r=1}^{J} \frac{K_r\gamma_r[-\gamma_r^{J+1}+(\alpha+1)\gamma_r^J-\alpha-\omega\gamma_r^J]}{[(N-1)\omega+1-\gamma_r]}$$

$$= \sum_{r=1}^{J} \frac{-\omega K_r\gamma_r^{J+1}}{(N-1)\omega+1-\gamma_r}\,, \tag{550}$$

in view of Equations (448,516).

Solution for n=S+2. We perform the same steps as for the case n=S+1. From (527), using (547) to express the right-hand side, we obtain

$$p_{S+2,j+1} - [(N-2)\omega+1]p_{S+2,j} = -(N-1)\omega p_{S+1,j}$$

$$= -(N-1)\omega p_o\{G_{S+1}[(N-1)\omega+1]^{j-1} + \sum_{r=1}^{J} A_{S+1,r}\gamma_r^j\}, \tag{551}$$

$$j=1,2,3,\cdots,J-1.$$

The homogeneous solution is

$$p_{S+2,j}^{(H)} = p_oG_{S+2}[(N-2)\omega+1]^{j-1}, \quad j=1,2,3,\cdots,J; \tag{552}$$

where G_{S+2} is independent of j. A particular solution is assumed in the form

$$p_{S+2,j}^{(P)} = p_o B_{S+2,S+1}[(N-1)\omega+1]^{j-1} + p_o \sum_{r=1}^{J} A_{S+2,r}\gamma_r^j, \tag{553}$$

where the undetermined coefficients $B_{S+2,S+1}$, $A_{S+2,1}$, $A_{S+2,2}$, $A_{S+2,3}$, $\cdots$, $A_{S+2,J}$ are independent of j. Substituting Equation (553) into (551), we find

$$p_o B_{S+2,S+1}[(N-1)\omega+1]^{j-1}\{[(N-1)\omega+1]-[(N-2)\omega+1]\}$$

$$+ p_o \sum_{r=1}^{J} A_{S+2,r}\gamma_r^j\{\gamma_r-[(N-2)\omega+1]\}$$

$$= -(N-1)\omega p_o G_{S+1}[(N-1)\omega+1]^{j-1} - (N-1)\omega p_o \sum_{r=1}^{J} A_{S+1,r}\gamma_r^j. \tag{554}$$

Equating appropriate coefficients, it follows that

$$B_{S+2,S+1} = -(N-1)G_{S+1}; \tag{555}$$

$$A_{S+2,r} = (N-1)\omega A_{S+1,r}[(N-2)\omega+1-\gamma_r]^{-1}, \quad r=1,2,3,\cdots,J. \tag{556}$$

The total solution is thus,

$$p_{S+2,j} = p_o G_{S+2}[(N-2)\omega+1]^{j-1} + p_o B_{S+2,S+1}[(N-1)\omega+1]^{j-1}$$

$$+ p_o \sum_{r=1}^{J} A_{S+2,r}\gamma_r^j, \quad j=1,2,3,\cdots,J; \tag{557}$$

where only p_o and G_{S+2} remain to be determined. The appropriate boundary condition for determining G_{S+2} is Equation (523) with n=S+1.

Solution for n=S+3. We perform the same steps as for the case n=S+1 and n=S+2. From (528), using (557) to express the right-hand side, we obtain

$$p_{S+3,j+1} - [(N-3)\omega+1]p_{S+3,j} = -(N-2)\omega p_{S+2,j}$$

$$= -(N-2)\omega p_o\{G_{S+2}[(N-2)\omega+1]^{j-1}$$

$$+ B_{S+2,S+1}[(N-1)\omega+1]^{j-1} + \sum_{r=1}^{J} A_{S+2,r}\gamma_r^j\}, \qquad (558)$$

$$j=1,2,3,\cdots,J-1.$$

The homogeneous solution is

$$p_{S+3,j}^{(H)} = p_o G_{S+3}[(N-3)\omega+1]^{j-1}, \quad j=1,2,3,\cdots,J; \qquad (559)$$

where G_{S+3} is independent of j.

A particular solution is assumed in the form

$$p_{S+3,j}^{(P)} = p_o B_{S+3,S+2}[(N-2)\omega+1]^{j-1} + p_o B_{S+3,S+1}[(N-1)\omega+1]^{j-1}$$

$$+ p_o \sum_{r=1}^{J} A_{S+3,r}\gamma_r^j, \qquad (560)$$

where the undetermined coefficients $B_{S+3,S+2}$, $B_{S+3,S+1}$, $A_{S+3,1}$, $A_{S+3,2}$, $A_{S+3,3}$, $\cdots$, $A_{S+3,J}$ are independent of j. Substituting

Equation (560) into (558), we find

$$p_oB_{S+3,S+2}[(N-2)\omega+1]^{j-1}\{[(N-2)\omega+1]-[(N-3)\omega+1]\}$$

$$+ p_oB_{S+3,S+1}[(N-1)\omega+1]^{j-1}\{[(N-1)\omega+1]-[(N-3)\omega+1]\}$$

$$+ p_o \sum_{r=1}^{J} A_{S+3,r}\gamma_r^j\{\gamma_r-[(N-3)\omega+1]\}$$

$$= -(N-2)\omega p_oG_{S+2}[(N-2)\omega+1]^{j-1} - (N-2)\omega p_oB_{S+2,S+1}[(N-1)\omega+1]^{j-1}$$

$$-(N-2)\omega p_o \sum_{r=1}^{J} A_{S+2,r}\gamma_r^j. \tag{561}$$

Equating appropriate coefficients, it follows that

$$B_{S+3,S+2} = -(N-2)G_{S+2}; \tag{562}$$

$$B_{S+3,S+1} = -(1/2)(N-2)B_{S+2,S+1}; \tag{563}$$

$$A_{S+3,r} = (N-2)\omega A_{S+2,r}[(N-3)\omega+1-\gamma_r]^{-1}, \quad r=1,2,3,\cdots,J. \tag{564}$$

The total solution is thus,

$$p_{S+3,j} = p_oG_{S+3}[(N-3)\omega+1]^{j-1} + p_oB_{S+3,S+2}[(N-2)\omega+1]^{j-1}$$

$$+ p_oB_{S+3,S+1}[(N-1)\omega+1]^{j-1} + p_o \sum_{r=1}^{J} A_{S+3,r}\gamma_r^j, \tag{565}$$

$$j=1,2,3,\cdots,J;$$

where only p_o and G_{S+3} remain to be determined. The appropriate boundary condition for determining G_{S+3} is Equation (523) with n=S+2.

Extrapolation to General n. Continuing the above procedure, we would find at the *kth* step a solution of the form

$$\begin{aligned} p_{S+k,j} = {} & p_o G_{S+k}[(N-k)\omega+1]^{j-1} + p_o B_{S+k,S+k-1}[(N-k+1)\omega+1]^{j-1} \\ & + p_o B_{S+k,S+k-2}[(N-k+2)\omega+1]^{j-1} \\ & + p_o B_{S+k,S+k-3}[(N-k+3)\omega+1]^{j-1} + \cdots \\ & + p_o B_{S+k,S+1}[(N-1)\omega+1]^{j-1} + p_o \sum_{r=1}^{J} A_{S+k,r}\gamma_r^j, \end{aligned} \tag{566}$$

$$j=1,2,3,\cdots,J.$$

We will now use an inductive approach to express the $B_{n,m}$ and $A_{n,r}$ in terms of known factors and/or the unknown coefficients G_n.

It can be seen from Equations (546,556,564) that we are generating a sequence of numbers $A_{S+k,r}$ according to the relations

$$A_{S,r} \equiv K_r; \tag{567}$$

$$A_{S+k,r} = \frac{(N-k+1)\omega A_{S+k-1,r}}{(N-k)\omega + 1 - \gamma_r}, \quad k=1,2,3,\cdots; \tag{568}$$

which may be expressed in the equivalent form

$$A_{S,r} \equiv K_r; \tag{569}$$

$$A_{n,r} = \frac{(N+S-n+1)\omega A_{n-1,r}}{(N+S-n)\omega + 1 - \gamma_r}, \quad n=S+1,S+2,S+3,\cdots. \tag{570}$$

Solving Equations (569-570) recursively, we find

$$A_{n,r} = \frac{N!\omega^{n-S}K_r}{(N+S-n)!} \prod_{m=S+1}^{n} [(N+S-m)\omega+1-\gamma_r]^{-1}, \tag{571}$$

$$n=S+1,S+2,S+3,\cdots.$$

From Equations (555,563), we see that we are generating a sequence of numbers $B_{S+k,S+1}$ according to the relations

$$B_{S+1,S+1} \equiv G_{S+1}; \tag{572}$$

$$B_{S+k,S+1} = \frac{-(N-k+1)}{(k-1)} B_{S+k-1,S+1}, \quad k=2,3,4,\cdots; \tag{573}$$

which may be expressed in the equivalent form

$$B_{S+1,S+1} \equiv G_{S+1}; \tag{574}$$

$$B_{n,S+1} = \frac{-(N+S-n+1)}{(n-S-1)} B_{n-1,S+1}, \quad n=S+2,S+3,S+4,\cdots. \tag{575}$$

Evaluating Equations (574-575) recursively, we find

$$B_{n,S+1} = \frac{(N-1)!(-1)^{n-S+1}G_{S+1}}{(N+S-n)!(n-S-1)!}, \quad n=S+2,S+3,S+4,\cdots. \qquad (576)$$

More generally, we suspect that

$$B_{n,m} = \frac{(N+S-m)!(-1)^{n-m}G_m}{(N+S-n)!(n-m)!}, \qquad (577)$$

for integral n and m such that $n>m\geq S+1$.

Accordingly, we are led to hypothesize that there exists a sequence of (arbitrary) numbers G_n ($n=S+1,S+2,S+3,\cdots,N+S$) such that the general solution to Equation (522) can be expressed in the form

$$p_{n,j} = p_{n,j}^{(H)} + p_{n,j}^{(P)} = p_o \sum_{m=S+1}^{n} \frac{(N+S-m)!(-1)^{n-m}G_m[(N+S-m)\omega+1]^{j-1}}{(N+S-n)!(n-m)!}$$

$$+ p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r^j\omega^{n-S}}{(N+S-n)!\prod_{m=S+1}^{n}[(N+S-m)\omega+1-\gamma_r]}, \qquad (578)$$

$$n=S+1,S+2,S+3,\cdots,N+S \quad \text{and} \quad j=1,2,3,\cdots,J;$$

where the component parts are

$$p_{n,j}^{(H)} = p_o G_n[(N+S-n)\omega+1]^{j-1}; \qquad (579)$$

$$p_{n,j}^{(P)} = p_o \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m}G_m[(N+S-m)\omega+1]^{j-1}}{(N+S-n)!(n-m)!}$$

$$+ p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r^j\omega^{n-S}}{(N+S-n)! \prod_{m=S+1}^{n}[(N+S-m)\omega+1-\gamma_r]} . \qquad (580)$$

The first summation in (580) is taken to be zero when $n=S+1$.

Verification of the General Solution. It has already been shown that (578) is the solution for $n=S+1, S+2$, and $S+3$. To establish the general validity of Equations (578-580) with respect to (522), it must be shown that

$$0 = p_{n,j+1}^{(H)} - [(N+S-n)\omega+1]p_{n,j}^{(H)}, \qquad (581)$$

$$0 = p_{n,j+1}^{(P)} - [(N+S-n)\omega+1]p_{n,j}^{(P)} + (N+S-n+1)\omega p_{n-1,j}, \qquad (582)$$

for arbitrary n,j in the domain of Equation (522).

Equation (581) is clearly satisfied by (579). Further, from Equations (578,580), we have

$$p_{n,j+1}^{(P)} - [(N+S-n)\omega+1]p_{n,j}^{(P)} + (N+S-n+1)\omega p_{n-1,j}$$

$$= \left\{ p_o \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m}G_m[(N+S-m)\omega+1]^j}{(N+S-n)!(n-m)!} \right.$$

$$+ p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r^{j+1}\omega^{n-S}}{(N+S-n)! \prod_{m=S+1}^{n} [(N+S-m)\omega+1-\gamma_r]} \Bigg\}$$

$$- [(N+S-n)\omega+1] \Bigg\{ p_o \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m}G_m[(N+S-m)\omega+1]^{j-1}}{(N+S-n)!(n-m)!}$$

$$+ p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r^{j}\omega^{n-S}}{(N+S-n)! \prod_{m=S+1}^{n} [(N+S-m)\omega+1-\gamma_r]} \Bigg\}$$

$$+ (N+S-n+1)\omega \Bigg\{ p_o \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m-1}G_m[(N+S-m)\omega+1]^{j-1}}{(N+S-n+1)!(n-m-1)!}$$

$$+ p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r^{j}\omega^{n-S-1}}{(N+S-n+1)! \prod_{m=S+1}^{n-1} [(N+S-m)\omega+1-\gamma_r]} \Bigg\}$$

$$= p_o \sum_{m=S+1}^{n-1} \Bigg\{ \left[\frac{(N+S-m)!(-1)^{n-m}G_m[(N+S-m)\omega+1]^{j-1}}{(N+S-n)!(n-m-1)!} \right]$$

$$\cdot \left[\frac{[(N+S-m)\omega+1]}{(n-m)} - \frac{[(N+S-n)\omega+1]}{(n-m)} + \frac{(N+S-n+1)\omega(-1)}{(N+S-n+1)} \right] \Bigg\}$$

$$+ p_o \sum_{r=1}^{J} \Bigg\{ \left[\frac{N!K_r\gamma_r^{j}\omega^{n-S-1}}{(N+S-n)! \prod_{m=S+1}^{n-1} [(N+S-m)\omega+1-\gamma_r]} \right]$$

$$\cdot \left[\frac{\gamma_r\omega}{[(N+S-n)\omega+1-\gamma_r]} - \frac{[(N+S-n)\omega+1]\omega}{[(N+S-n)\omega+1-\gamma_r]} + \frac{(N+S-n+1)\omega}{(N+S-n+1)}\right]\Bigg\}$$

$$= 0, \tag{583}$$

since the second factor in each term vanishes independently. Thus, Equation (582) is also satisfied. It follows that the general solution of Equation (522) is given by Equation (578).

Determination of the G_n. The value of G_{S+1} has already been obtained and recorded in Equation (550). To determine G_{S+2}, we write the boundary condition (523), with n=S+1, in the form

$$0 = p_{S+2,1} - [(N-1)\omega+1]p_{S+1,J} + N\omega p_{S,J}, \tag{584}$$

and substitute from Equations (538,578) to find

$$\begin{aligned} 0 = \{&p_oG_{S+2} - p_oG_{S+1}(N-1) \\ &+ p_o\sum_{r=1}^{J} N(N-1)K_r\gamma_r\omega^2[(N-1)\omega+1-\gamma_r]^{-1}[(N-2)\omega+1-\gamma_r]^{-1}\} \\ &- [(N-1)\omega+1]\ \{p_oG_{S+1}[(N-1)\omega+1]^{J-1} \\ &+ p_o\sum_{r=1}^{J} NK_r\omega\gamma_r^J[(N-1)\omega+1-\gamma_r]^{-1}\} + N\omega\{p_o\sum_{r=1}^{J} K_r\gamma_r^J\}. \end{aligned} \tag{585}$$

Hence, with the aid of (448,516), we have

$$G_{S+2} = G_{S+1}\{(N-1) + [(N-1)\omega+1]^J\}$$

$$+ \sum_{r=1}^{J} \left\{\frac{N\omega^2 K_r\gamma_r(1-2\gamma_r^J)}{[(N-1)\omega+1-\gamma_r][(N-2)\omega+1-\gamma_r]}\right\}, \tag{586}$$

where K_r and G_{S+1} are given by Equations (537,550).

To determine the remaining G_n, we write Equations (523) in the form

$$0 = p_{n,1} - [(N+S-n+1)\omega+1]p_{n-1,J} + (N+S-n+2)\omega p_{n-2,J}, \tag{587}$$

$$n=S+2,S+3,S+4,\cdots,N+S;$$

and substitute from Equation (578) to obtain

$$0 = \left\{p_o \sum_{m=S+1}^{n} \frac{(N+S-m)!(-1)^{n-m}G_m}{(N+S-n)!(n-m)!} + p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r\omega^{n-S}}{(N+S-n)! \prod_{m=S+1}^{n}[(N+S-m)\omega+1-\gamma_r]}\right\}$$

$$- [(N+S-n+1)\omega+1]\left\{p_o \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m-1}G_m[(N+S-m)\omega+1]^{J-1}}{(N+S-n+1)!(n-m-1)!}\right.$$

$$\left. + p_o \sum_{r=1}^{J} \frac{N!K_r\gamma_r^J\omega^{n-S-1}}{(N+S-n+1)! \prod_{m=S+1}^{n-1}[(N+S-m)\omega+1-\gamma_r]}\right\}$$

$$+ (N+S-n+2)\omega \left\{ p_o \sum_{m=S+1}^{n-2} \frac{(N+S-m)!(-1)^{n-m-2} G_m [(N+S-m)\omega+1]^{J-1}}{(N+S-n+2)!(n-m-2)!} \right.$$

$$\left. + \; p_o \sum_{r=1}^{J} \frac{N! K_r \gamma_r^J \omega^{n-S-2}}{(N+S-n+2)! \prod_{m=S+1}^{n-2} [(N+S-m)\omega+1-\gamma_r]} \right\} , \qquad (588)$$

$$n=S+3, S+4, S+5, \cdots, N+S.$$

Hence, transposing G_n to the left side and using Equations (448,516) to simplify expressions involving the γ_r, we obtain the recurrence relation

$$G_n = \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m+1} G_m}{(N+S-n)!(n-m)!} \left[1 + \frac{(n-m)[(N+S-m)\omega+1]^J}{(N+S-n+1)} \right]$$

$$+ \sum_{r=1}^{J} \frac{N! K_r \gamma_r \omega^{n-S} [(n-S)(1-\gamma_r^J)-1]}{(N+S-n+1)! \prod_{m=S+1}^{n} [(N+S-m)\omega+1-\gamma_r]} , \qquad (589)$$

$$n=S+3, S+4, S+5, \cdots, N+S;$$

where K_r, G_{S+1}, and G_{S+2} are given by Equations (537,550,586). However, comparison of Equations (586,589) shows that $n = S + 2$ may also be included in the domain of Equation (589). Further, if the convention is adopted that undefined summations vanish, then Equation (589) suffices also to specify G_{S+1}.

The ratios $p_{n,j}/p_o$ are now known for all n and j in the problem domain. In this case, the unused boundary condition (524) is redundant as can be shown by substituting (578) into (524) and using (448,516,589) to establish an identity. The computations involved are quite lengthy and not intrinsically rewarding; hence, they will not be recorded here.

Results Collected

The principal results of this section are given by Equations (451,516,525,537,550,578,586,589). In summary, for

$$J\alpha = JN\omega = (N\lambda/\mu) < 1, \tag{590}$$

the stationary probabilities are

$$\left.\begin{aligned}
p_{n,j} &= p_o \sum_{r=1}^{J} K_r \gamma_r^{J(n-S)+j}, \\
&\qquad\qquad n=1,2,3,\cdots,S \text{ and } j=1,2,3,\cdots,J; \\
p_{n,j} &= p_o \sum_{m=S+1}^{n} \frac{(N+S-m)!(-1)^{n-m} G_m [(N+S-m)\omega+1]^{j-1}}{(N+S-n)!(n-m)!} \\
&\quad + p_o \sum_{r=1}^{J} \frac{N! K_r \gamma_r^j \omega^{n-S}}{(N+S-n)! \prod_{m=S+1}^{n} [(N+S-m)\omega+1-\gamma_r]}, \\
&\qquad n=S+1,S+2,S+3,\cdots,N+S \text{ and } j=1,2,3,\cdots,J;
\end{aligned}\right\} \tag{591}$$

where the γ_r $(r=1,2,3,\cdots,J)$ are the necessarily-distinct roots of the real polynomial*

$$\gamma^J - \alpha(\gamma^{J-1}+\gamma^{J-2}+\gamma^{J-3}+\cdots+\gamma+1) = 0; \tag{592}$$

where

$$K_r = \frac{(1-\gamma_r)\gamma_r^{J(S-1)}}{[J(\alpha+1)-(J+1)\gamma_r]}, \quad r=1,2,3,\cdots,J; \tag{593}$$

where the G_n are defined recursively by

$$\left.\begin{aligned} G_{S+1} &= -\sum_{r=1}^{J} \frac{\omega K_r \gamma_r^{J+1}}{[(N-1)\omega+1-\gamma_r]}; \\ G_n &= \sum_{m=S+1}^{n-1} \frac{(N+S-m)!(-1)^{n-m+1}G_m}{(N+S-n)!(n-m)!}\left[1+\frac{(n-m)[(N+S-m)\omega+1]^J}{(N+S-n+1)}\right] \\ &\quad + \sum_{r=1}^{J} \frac{N!K_r\gamma_r\omega^{n-S}[(n-S)(1-\gamma_r^J)-1]}{(N+S-n+1)!\prod_{m=S+1}^{n}[(N+S-m)\omega+1-\gamma_r]}, \\ &\qquad n=S+2,S+3,S+4,\cdots,N+S; \end{aligned}\right\} \tag{594}$$

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*Equation (592) previously appeared as Equation (451). Some properties of the roots of this characteristic equation are given by Equations (449-458).

and where p_o is to be determined from the normalizing condition,

$$1 = p_o + \sum_{n=1}^{N+S} \sum_{j=1}^{J} p_{n,j}. \tag{595}$$

The Unconditional State Probabilities

The probabilities p_n, of there being n failed machines in the system irrespective of the service phase, are obtained by summing the $p_{n,j}$ over all possible phases $j=1,2,3,\cdots,J$. Noting the occurrence of geometric progressions in (591), we have

$$\left.\begin{aligned}
p_n &= p_o \sum_{r=1}^{J} \{\gamma_r^{J(n-1)+1}(1-\gamma_r^J)[J(\alpha+1)-(J+1)\gamma_r]^{-1}\}, \\
&\qquad\qquad n=1,2,3,\cdots,S; \\
p_n &= p_o \sum_{\substack{m=S+1 \\ m\neq N+S}}^{n} \frac{(N+S-m-1)!(-1)^{n-m+1}G_m\{1-[(N+S-m)\omega+1]^J\}}{(N+S-n)!(n-m)!\omega} \\
&+ p_o \sum_{r=1}^{J} \frac{N!\omega^{n-S}\gamma_r^{J(S-1)+1}(1-\gamma_r^J)[J(\alpha+1)-(J+1)\gamma_r]^{-1}}{(N+S-n)! \prod_{m=S+1}^{n} [(N+S-m)\omega+1-\gamma_r]}, \\
&\qquad\qquad n=S+1,S+2,S+3,\cdots,N+S;
\end{aligned}\right\} \tag{596}$$

where the restrictions and definitions (590,592,594) are applicable and where p_o is to be determined from the normalizing condition

$$1 = \sum_{n=0}^{N+S} p_n. \tag{597}$$

Extension to More Complex Models

The initial step in each of the model derivations of this chapter was to use the "method of successive stages" to partition chi-square distributed* service times into a set of successive phases, each having a negative-exponentially distributed execution time. This artifice permitted the application of simple Poisson queueing theory in the formulation of (partitioned) equations of state for decidedly non-Poisson processes. In each case, it was assumed that the machine failure times were negative exponential variates and there was but a single server.

The method of successive stages can be applied to more complex repairman models with spares. Basically, the principle is to partition *every* chi-square distributed interoccurrence phenomenon into discrete phases, each phase requiring a negative-exponentially distributed execution time. It was seen earlier in the chapter how the method applies to the partitioning of service times. For chi-square failure time distributions, the rationale is much the same. A machine would be considered to begin failing at the instant it is placed into operation. The complete failure process would be regarded as occurring in, say L, discrete phases, each phase requiring a negative-exponentially distributed execution time. Of course, an extra variable, say ℓ, would have to be added to the state descriptor to keep track of the current

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*The term "chi-square distribution" as used in this section is intended only to include scale-modified chi-square distributions, with even degrees of freedom, of the type shown in Equation (342).

failure status of the working machine.*

The method of successive stages is easily applied to the *formulation* of state equations for some very general queueing models with chi-square distributed interoccurrence times. The difficulty, which is a serious one, lies in the subsequent task of *solving* the resulting mass of multivariate equations. For example, a model with N machine working positions, c servers, and chi-square distributed failure and service times might require as many as $N + c + 1$ variables to keep track of the number of units in the service queue, the current phase of service at each channel, and the current phase of failure at each working position. Of course, some simplification may be possible if the service (failure) times at each channel (working position) follow the same chi-square distribution. In this event, it would only be necessary to keep track of the number of units currently in the j*th* service (ℓ*th* failure) phase. A total of $L + J + 1$ variables would be needed.

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*If this modification were made to one of the models of this chapter for the case of one working position ($N=1$), the states of the elementary Markov chain would be described by the triplets (n,j,ℓ).

CHAPTER VI

DISCUSSION, CONCLUSIONS, AND RECOMMENDATIONS

The Investigation in Retrospect

This investigation has been concerned with the extension of the bounds of knowledge relating to the design of maintenance facilities and spares-stocking policies for systems which are essentially characterized by the use of rotatable, fleet-operated components. The general research objective has been the development of a theoretical basis for the representation and study of congestion phenomena associated with the flow of nonoperative components through a limited-capacity service facility and their subsequent storage before being returned to use. Pursuit of this objective led to a number of interesting--and occassionally unexpected--results.

The path of study to the development of "repairman models with spares" has proceeded through several areas of the queueing theory literature--repairman models, cyclic queues, finite queues, networks of queues, etc.--and involved a number of related investigative techniques. An initial concern was the finding of a methodological background to supplement the meager offerings of the few papers identifiable as treating repairman problems with spares. It was found that the literature on repairman models, the direct historical and natural methodological precursors to repairman models with spares,* provides a

*It was in fact this relationship that prompted the name rich source of theory, analytic techniques, and procedural philosophy of potential usefulness to any study in the realm of repairman models with spares.

By contrasting the relative bounds of knowledge on regular repairman models (Chapter II) and those with spares (Chapter III), it was possible to obtain a measure of the relative level of development of theory on repairman models with spares. The existing repairman models with spares, namely the models of Taylor and Jackson [232] and Toft and Boothroyd [240], were seen to be analogous to the original repairman model devised by Palm [173] in 1947. Even the limited level of theoretical sophistication suggested by these models was diminished when it was shown that both models (and Palm's model as well) follow readily from an appropriate choice of parameters in the equations (10-11) of the general birth-and-death model of stochastic-processes theory.

Another benefit of the contrast was an indication, by analogy, of unsolved repairman problems with spares for which solutions would be desirable, welcome, and worthwhile. A number of these models were derived in the present investigation; however, many more remain to be developed. The study of repairman problems with spares may be regarded as a nearly virginal area of profitable research within the general realm of queueing theory applications.

The lessons learned from this contrast of regular repairman models and those with spares were applied to good advantage in the development of new repairman models with spares in Chapter IV. The

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"repairman models with spares."

principal results are outlined in Table 1. These models will be discussed below with emphasis on their general relevance to the study of repairman models with spares. More specific comments on the attributes of individual models were given at the time these models were presented and will not be repeated here.

The general birth-and-death equations (10-11) were applied, with an appropriate choice of parameters, to the development of models involving spares which could fail during storage (Table 1, Nos. 3-5). The models were shown to include Taylor and Jackson's and Toft and Boothroyd's models as special instances. The models obtained are, of course, just one possible application of the general birth-and-death equations. Many more repairman models with spares can be developed in the same way since the equations permit specification of the arrival and service rates, λ_n and μ_n, as functions of the system's state variable n.*

In terms of results obtained, the most valuable formulation technique for Poisson models, that is, for models with negative-exponentially distributed interoccurrence times, proved to be that of relating the system's state probabilities of times $t + \Delta t$ and t through a consideration of the possible happenings during the small interval Δt. The resulting expression, when evaluated as $\Delta t \to 0$, yields the dynamic equations governing the system, and, from these, the stationary equations

*In the case of infinite-waiting-line queueing models, this flexibility of representation has been used to depict such interesting phenomena as balking, truncated waiting lines, arrival discouragement, immediate servicing, and other forms of adaptive arrival and servicing behavior.

Table 1. Outline of Principal Results: A Guide to Repairman Models with Spares Treated in the Text

No.	Chapter, Section Title, Characterization	Type*	Formulation	Transition Rates** (λ_n, μ_n)	Numbers of Servers	Total Units in System	Number of Spares	Number of Operating Positions	Dynamic State Equations	Stationary State Equations	Dynamic Solution	Stationary Solution	Remarks, Significant Subcases
1.	Taylor and Jackson's model [232], reviewed in Chapter III	I	Adaptation of General Birth-and-Death model shown in Chapter III (cf. method in [232])	Equations (40)	c	N+S	S	N	Equations (6-8)	Equations (9)	-	Equations (41-42)	One of two pre-existing repairman models with spares. May be obtained as a subcase of the general birth and death model (equations (6-11)).
2.	Toft and Boothroyd's model [240], reviewed in Chapter III	II	Adaptation of General Birth-and-Death model shown in Chapter III (cf. method in [240])	(57-58)	c	N+S	S	N	(6-8)	(9)	-	(59-61)	One of two pre-existing repairman models with spares. May be obtained as subcase of birth-and-death model. May also be obtained from adaptation of Koenigsburg's two-stage cyclic queueing model [139] (271-277) as shown in Chapter IV.
3.	IV. "Models with Spares which Fail in Storage"--Units in spares inventory fail negative-exponentially with mean rate $\gamma\lambda$, $0 \le \gamma \le 1$, where λ is mean failure rate of an operating unit.	I	Birth and Death	(67)	c	N+S	S	N	(6-8)	(9)	-	(68-71)	With $\gamma=0$, have Taylor and Jackson's model. With $\gamma=1$, have Palm's model [173-174] of repairmen without spares.
4.		II $S \ge c$	Birth and Death	(74)	c	N+S	S	N	(6-8)	(9)	-	(75-76), (69-70)	With $\gamma=0$, have Toft and Boothroyd's model. With $\gamma=1$, have the specialized model (76-77).
5.		II $S < c$	Birth and Death	(78)	c	N+S	S	N	(6-8)	(9)	-	(79,76), (69-70)	With $\gamma=0$, have Toft and Boothroyd's model. With $\gamma=1$, have the specialized model (80,76).
6.	IV. "A Model with Extended Service Intervals"--After a system breakdown, an extended service interval ensues in which all but x $(0 \le x < S)$ units are made operative. The system is then restarted (with S-x standby spares available)	I $x \ge c$	Constructed from consideration of the possible happenings during a small interval Δt as $\Delta t \to 0$	-	c	N+S	S	N	(82-86), (89-90)	(91-98), (116-122)	-	(111-115)	With x=S, have Taylor and Jackson's model.
7.		I $x < c$	Events during Δt as $\Delta t \to 0$	-	c	N+S	S	N	(82-88), (90)	(123-131), (158-165), (166-172)	-	(134-135), (157), (149-150), (155),(115)	

*Type I models--The system breaks down when there are less than N operative units and is not restarted until servicing restores the operative number to N.
Type II models--The system operates at reduced capacity when there are less than N operative units.

**A unit's failure and service times are negative-exponential variates (see Equation (2)) in models 1-14. In models 15-16, failure times are negative-exponential variates; however, service times follow a chi-square distribution (see Equation (342)) with even degrees of freedom. These latter service times are also referred to as Erlang, gamma, or Pearson type-III variates.

Table 1. Outline of Principal Results: A Guide to Repairman Models with Spares Treated in the Text (Continued)

No.	Chapter, Section Title, Characterization	Type*	Formulation	Transition Rates** (λ_n, μ_n)	Number of Servers	Total Units in System	Number of Spares	Number of Operating Positions	Dynamic State Equations	Stationary State Equations	Dynamic Solution	Stationary Solution	Remarks, Significant Subcases
8.	IV. "A Time-Dependent Model"—The time-dependent version of Taylor and Jackson's model with one server	I	Birth and Death	(40) c→1 S→W	1	N+W	W	N	(173-174)	(9)	(199), (203), (182-183), (212-215)	(41-42)	The dynamic solution is obtained in the Laplace-transform domain. Translation to the time-domain is indicated.
9.	IV. "A Model Involving Ancillary Duties"--Intervals separating calls for ancillary duty and durations of ancillary duty are negative-exponentially distributed	I	Equations (216-221); events during Δt as Δt→0	(40) c→1 augmented	1	N+S	S	N	(222-227)	(228-233)	-	(243-244), (246-247), (253-254), (234)	Process is like that studied by Taylor and Jackson but has the additional ramification of ancillary duties.
10.	IV. "Models Involving Faulty Repairs"--A newly repaired unit immediately fails with probability α (0≤α<1) and re-enters the repair queue. The process is envisioned as repair followed by Yes-No quality control testing	I	Equations (256-262); events during Δt as Δt→0. Later shown to be subcase of Birth-and-Death model	(263)	c	N+S	S	N	(6-8)	(9)	-	(264-265)	Model is analogous to a two-stage cyclic queue with self-feedback at the service stage. Model is found to be equivalent to Taylor and Jackson's model with mean service rate of a repairman reduced by the factor (1-α). It is demonstrated that total time to successfully restore a unit to serviceability is a negative-exponential variate with mean 1/[(1-α)μ].
11.		II	Birth-and-Death (by analogy with model no. 10)	(57-58) μ→(1-α)μ	c	N+S	S	N	(6-8)	(9)	-	(269-270), (59-61)	Model is equivalent to Toft and Boothroyd's model with mean service rate reduced by the factor (1-α). Total time to successfully restore a unit to serviceability is a negative-exponential variate with mean 1/[(1-α)μ].
12.	IV. "Models with Transit Delays"--Units suffer negative-exponentially distributed transit delays in moving from operating positions to the repair facility and vice versa.	II	Adaptation of Pelczynski's model [177] of a closed network of $M \mid M \mid c$ stations with stochastic transitions from station i to station j	(281-286)	c	N+S in 4 possible locations	S	N	-	(287-293)	-	(305-307), (310-312)	Solution makes use of Pelczynski's model which is elucidated in Equations (287-290), (294) of Chapter IV. The system is like that studied by Toft and Boothroyd, but the assumption of instantaneous unit movements is replaced with that of negative-exponentially distributed transit times.

*Type I models--The system breaks down when there are less than N operative units and is not restarted until servicing restores the operative number to N.
Type II models--The system operates at reduced capacity when there are less than N operative units.

**A unit's failure and service times are negative-exponential variates (see Equation (2)) in models 1-14. In models 15-16, failure times are negative-exponential variates; however, service times follow a chi-square distribution (see Equation (342)) with even degrees of freedom. These latter service times are also referred to as Erlang, gamma, or Pearson type-III variates.

Table 1. Outline of Principal Results: A Guide to Repairman Models with Spares Treated in the Text (Continued)

No.	Chapter, Section Title, Characterization	Type*	Formulation	Transition Rates** (λ_n, μ_n)	Number of Servers	Total Units in System	Number of Spares	Number of Operating Positions	Dynamic State Equations	Stationary State Equations	Dynamic Solution	Stationary Solution	Remarks, Significant Subcases
13.	IV. "Models with Transit Delays"--Units suffer generally distributed transit delays in moving from operating positions to the repair facility and vice versa.	II	Adaptation of Posner and Bernholtz's cyclic queueing model [190] with generally distributed inter-stage transit times	-	c	M in four possible locations	M-N	N	-	-	-	(318-321), (306-307)	
14.	IV. "A Problem in Servicing Aircraft Engines"--Units fail in two possible ways. After inspection, failed units go to one of two discrete repair facilities according to the failure mode.	II	Adaptation of Pelczynski's model	(322-330)	c_2, c_3	M in three possible locations	M-N	N	-	(287-290)	-	(337-341)	Although this model is a repairman model with spares via interpretation, it is more properly classified as a "finite queue" model.
15.	V. "The Type I Model with Chi-Square Servicing"	I	Equations (348-355); events in Δt as $\Delta t \to 0$. "Method of stages"	(347) Markov process	1	N+S	S	N	(357-363)	(371-378)	-	(484-485), (451), (486,488)	With two degrees of freedom, the model specializes to Taylor and Jackson's single-server model.
16.	V. "The Type II Model with Chi-Square Servicing"	II	Equations (490-499); events in Δt as $\Delta t \to 0$. "Method of stages"	(489) Markov process	1	N+S	S	N	(500-507)	(516-524)	-	(590-597), (451)	With two degrees of freedom, the model specializes to Toft and Boothroyd's single-server model.

*Type I models--The system breaks down when there are less than N operative units and is not restarted until servicing restores the operative number to N.

Type II models--The system operates at reduced capacity when there are less than N operative units.

**A unit's failure and service times are negative-exponential variates (see Equation (2)) in models 1-14. In models 15-16, failure times are negative-exponential variates; however, service times follow a chi-square distribution (see Equation (342)) with even degrees of freedom. These latter service times are also referred to as Erlang, gamma, or Pearson type-III variates.

easily follow. The procedure was applied in Chapter IV, where it was used to formulate repairman models with spares involving extended service intervals (Table 1, Nos. 6-7), ancillary duties (No. 9), and faulty repairs (Nos. 10-11) and in Chapter V, where it was applied through a "method of stages" to the formulation of models involving chi-square distributed service times (Table 1, Nos. 15-16).

This approach, which will undoubtedly play an important role in future investigations of repairman problems with spares, is in fact the first formal, and now fundamental, model formulation technique of the theory of Poisson queues. It was the technique which Palm, Taylor and Jackson, Toft and Boothroyd, and many other researchers (surveyed in Chapters II and III) used to formulate their models. Indeed, by tracing back to original sources in the literature, it will be found that the technique underlies every model formulation scheme used in the present investigation.

One key to the successful and valid application of this fundamental formulation method of Poisson queues is the selection of an appropriate set of states of the system among which transitions will follow a Markov process. In some cases, this step can be accomplished in a straightforward manner; in others, some device such as the so-called "supplementary variable technique" must be employed to ensure that the chosen states of the system will form a Markov chain. For example, in order to formulate the repairman models with spares of Chapter IV which involve extended service intervals (Table 1, Nos. 6-7), it was necessary to include a number of supplementary servicing states which themselves had no relevant physical meaning, but which permitted all transition times to occur according to the memoryless (Markovian) negative exponential distribution. This device permitted the representation of artificially extended (e.g., via policy) service intervals so that one could address through model manipulation such considerations as the appropriate number of workable spares to accumulate before restarting a failed system.

The fundamental formulation technique does not have to be confined to simple failure/repair models. For example, it may be applied to the formulation of models involving several independent arrival sources as was shown in Chapter IV in the derivation of repairman models with spares in which demands for ancillary work arrived at negative-exponentially distributed intervals and lasted for periods of time that were also negative-exponentially distributed (Table 1, No. 9). Further, time- and state-independent stochastic phenomena may be included in such formulations. For example, in Chapter IV, repairman models with spares, in which a newly repaired unit immediately fails with probability α ($0 \le \alpha < 1$), were created by simply partitioning the service rate μ into components $\alpha\mu$ and $(1-\alpha)\mu$ and introducing the notion of virtual transitions from a particular state to itself (Table 1, Nos. 10-11).

The fundamental formulation technique of Poisson queues is a powerful and valuable tool for the development of equations of state for Poisson repairman models with spares. However, the technique is only a formulation scheme. It is entirely a different matter to solve the resulting set of differential-difference equations relating the dynamic probabilities, or the set of difference equations relating the stationary probabilities, in order to obtain explicit expressions for the state probabilities. As discussed *passim* in Chapters IV and V, the existing techniques for solving difference equations are severely limited in scope. This limitation is probably the primary obstacle that will have to be overcome in future investigations of Poisson repairman problems with spares.

Relative to the potential difficulty of solving equations of state, it should be observed that steady-state solutions are invariably easier to obtain than are dynamic solutions. One demonstration of this fact appears in Chapter IV where a single-server, time-dependent analogue to Taylor and Jackson's model was obtained in Laplace-transform domain only after some laborious computations (Table 1, No. 8). The analogous multiserver, stationary model was obtained by Taylor and Jackson from a straightforward induction and in the present investigation by simply making an appropriate substitution into the equations (10-11) of the general birth-and-death model. It may be concluded that steady-state results will predominate in future investigations of repairman models with spares just as they have in the present investigation.

The theoretical background supporting the study of repairman models with spares was further augmented in the latter sections of Chapter III with the introduction of topics from queueing network theory. In ascending order of complexity and generality, the hierarchy of models discussed was: (1) repairman models, (2) repairman models with spares, (3) cyclic queues, (4) finite queues, and (5) networks of queues. Conceptually, each group of models may be viewed as being generalizations of models lower in the hierarchy and as being specializations of models higher in the hierarchy.

The existence in the literature of this hierarchy of queueing models has important implications on the study of repairman models with spares. In general, it will be found that the literature on regular repairman models and, to a lesser extent, that on cyclic queues offers a background of theory, methodology, and procedural philosophy for the study of new repairman models with spares. On the other hand, while the literature on cyclic queues, finite queues, and networks of queues does make important contributions to this background, it must be viewed primarily as a source of existing models to be specialized to the representation of repairman systems with spares.

The existing literature on cyclic and finite queues was applied to the development of repairman models with spares in the latter part of Chapter IV. Of primary importance was the establishment of a conceptual basis for relating the repairman-model-with-spares viewpoint of a single service center fed by a finite arrival source and the cyclic-queues viewpoint of two service centers, each with an attendant waiting line, visited in rotation by a continuously circulating group of units. This conceptual relationship was discussed qualitatively in Chapter III, and quantitatively in Chapter IV where a demonstration was given of the equivalence between the repairman model with spares of Toft and Boothroyd [240] and the two-stage cyclic queue of Koenigsberg [139].

The most useful result from cyclic and finite queueing theory proved to be Pelczynski's general model [177] for multistation finite Poisson queues (see Equations (287-290,294-295)). In one case in Chapter IV, his model was specialized to the form of a four-stage cyclic queue which was then adapted to the representation of a repairman model with spares in which units suffered transit delays in being transported to and from the service center (Table 1, No. 12). Another application of Pelczynski's model was to the development of a model for the servicing of aircraft engines by a commercial airline. This last model demonstrated by example the means for developing repairman models with spares in which there are multiple failure modes and/or multiple service centers (Table 1, No. 14).

With respect to future investigations of repairman problems with spares, the extent of potential benefit from existing cyclic and finite queueing theory should not be overestimated. The majority of existing cyclic and finite queueing models did not prove useful in the present investigation. There were three principal reasons for this: When evaluated for the case of only two stations in cyclic order, i.e., the base case for repairman models with spares, most such models had to be discarded because they either (1) became identical to Koenigsberg's model and hence duplicated Toft and Boothroyd's results; (2) reduced to a trivial model involving two single-server stages (i.e., one repairman and one machine working position); or (3) included phenomena (e.g., exogenous arrivals) inconsistent with the conceptualization of a repairman system with spares.

In Chapter IV, analysis was confined to problems with service times following the one-parameter (μ) negative exponential distribution. In Chapter V, the scope of potential applications of repairman models with spares was considerably broadened with the development of models with service times following the two-parameter (μ,J) chi-square distribution with even (2J) degrees of freedom, also known as the "*Jth* Erlang distribution" (Table 1, Nos. 15-16). Extensions to more general models were discussed qualitatively. Contrary to initial expectations, the expressions obtained for state probabilities were relatively simple in form and content. However, in each model, the solution was found to depend upon the roots of a *Jth* degree polynomial (Equation (451)). Thus, for large values of J, applications of the models would probably be impossible without the services of a digital or analogue computer to determine the roots of this polynomial.

Recommendations for Further Study

Implementation of Solutions

In the models of Chapter IV, most of the expressions obtained for state probabilities p_n are quite tractable to hand computations with the aid of standard mathematical tables. However, should one envision extensive use of a model or wish to apply it to a system having a large number of possible states, the development of tabled solutions is recommended for the purpose of generally facilitating computations. For any of the models of Chapter IV, a digital computer could easily be programmed to generate tables of state probabilities for a selected range of values of the relevant parameters (e.g. λ/μ, c, N, S).

The essential aspects of such a computer program would be a general formula (e.g., as a subroutine) for the ratios p_n/p_o and a variable, say R, to accumulate the sum of the p_n/p_o values. Core storage requirements would be minimal, even for large problems, since the p_n/p_o values could be output on magnetic tape as they were created and later read in a few at a time to be divided by the normalizing factor (i.e., the final value of R) just before being printed.

With respect to implementation of the Chapter V models having chi-square distributed service times with 2J degrees of freedom, there is a prerequisite need for solutions of the characteristic equation (451). In the case of large J, the roots of the characteristic equation will undoubtedly have to be obtained numerically using one of the standard methods--e.g., Bernoulli's iteration, Graeffe's root-squaring technique, Lin's iteration, Bairstow's method--for specified values of the parameters $N\lambda/\mu$ and J.

Since this characteristic equation arises in a variety of problems involving chi-square distributed interoccurrence times (see Syski [212], pp.310-328) or involving bulk arrivals and servicing (see Syski [212], pp.554-558), a very worthwhile project would be the development and publishing of tables of roots of the characteristic equation for a range of values of $N\lambda/\mu$ and J. Although the basic computer program would have to be quite lengthy to accommodate computations involving the real and imaginary parts of complex variables, it is suggested that the programming effort required to calculate these roots on a digital computer would be well justified. Computations could be simplified somewhat by using the augmented characteristic equation (448) and later discarding the spurious root $\gamma = 1$.

Application of Models

The present investigation has resulted in the derivation of symbolically general models for a variety of repairman systems with spares. Unfortunately, the forms in which expressions for the state probabilities were obtained did not prove favorable to the development of general formulae for model statistics or to any other type of model exploitation in general symbolic terms.* Since the full investigative and planning potential of the models cannot be realized without some knowledge of measures of system effectiveness, it is recommended that numerical work, using a digital computer, be instigated to develop tables of relevant statistics for each of the models of Chapters IV and V. This work could be perhaps most advantageously performed in conjunction with the solution-implementation projects suggested in the preceding section.

A useful, yet concise, set of parameter values upon which to base the tables would be the integers from 1 to 500 (or higher) for the integer-valued parameters (e.g., S, N, N+S, c, J) and a set of values permitting accurate linear interpolation for noninteger-valued parameters (e.g., λ/μ, $N\lambda/\mu$, $N\lambda/(J\mu)$). The sensitivity analysis that would be required to determine the minimal set of parameter values permitting valid linear interpolation for nontabled cases would itself be of interest.

New Theoretical Studies

The results of the present investigation represent only a beginning toward the development of a comprehensive knowledge of repairman

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*Actually, as discussed in the introduction to Chapter IV, there is no difficulty in writing explicit, symbolically-general formulae for model statistics. This may be done by merely substituting the state-probability expressions into the equations defining the statistics (e.g., Equations (43-47,62-63) with upper summation limits adjusted to the particular model). However, due to an inability to algebraically combine terms, the resulting expressions will be too complex to have any practical significance.

models with spares. For the important applications to the design of service facilities and spares-stocking policies for fleet-operated systems, both refinements and extensions of existing theory are needed. As a starting point, it is suggested that research be initiated with the general goal--but not the bound--of bringing the level of development of theory on repairman models with spares up to that extant for regular repairman models. Some specific recommendations for further work are:

1. Development of models which include several of the phenomena--e.g., extended service intervals, ancillary duties, transit delays, faulty repairs, etc.--that were represented in separate models in the present investigation.

2. Development of multiserver models with chi-square distributed failure and service times as extensions to the models of Chapter V.

3. Development of time-dependent models for more complicated repairman systems with spares than that considered in Chapter IV.

4. Development of models involving general, arbitrary interoccurrence-time distributions.

5. Development of models involving priority servicing.

APPENDIX A

THE METHOD OF VARIATION OF PARAMETERS

Introduction

Several derivations in the main body of the dissertation make reference to the "method of variation of parameters" for obtaining particular solutions of certain linear difference equations. The method is usually presented in the literature as a device for treating linear differential equations, but it has an analogous application in the solution of linear difference equations.

This appendix will review the underlying rationale of the method as applied to the general linear difference equations of the first and second orders. For examples and additional material on the solution of difference equations, reference is made to Chapter 3 of Hildebrand's book [83].

Pertinent to the presentation is a knowledge of the various forms in which one may express the solution to the difference equation

$$\Delta x_{n-1} \equiv x_n - x_{n-1} = \nu_n, \tag{598}$$

where Δ is the usual forward difference operator and ν_n is a known function of n. Writing $x_o = \eta$, we can determine successively from (598),

$$x_1 = \eta + \nu_1; \tag{599}$$

$$x_2 = x_1 + \nu_2 = \eta + \nu_1 + \nu_2; \tag{600}$$

$$x_3 = x_2 + \nu_3 = \eta + \nu_1 + \nu_2 + \nu_3; \cdots; \tag{601}$$

so that, by induction, we have

$$x_n = \eta + \sum_{k=1}^{n} \nu_k. \tag{602}$$

It can be seen that the arbitrary constant η is the homogeneous solution $x_n^{(H)}$ of Equations (598); that is, the solution of (598) for the case $\nu_n \equiv 0$. Further, noting that a change in the lower limit of the summation can be counteracted by a change in the value of η, it follows that a particular solution $x_n^{(P)}$ of Equations (598) is given by

$$x_n^{(P)} = \sum_{k=L}^{n} \nu_k \tag{603}$$

for an arbitrary choice of L. Since Equations (603) are defined only for $n \geq L$, practical considerations usually motivate the choice of L as being the smallest value from the range of n that is of interest in a specific problem.

The General Linear Difference Equation of First Order

The general linear difference equation of first order may be written in the form

$$Tx_n \equiv a_n x_n + b_n x_{n-1} = \xi_n, \tag{604}$$

where T is a linear operator defined by the equation and a_n, b_n, and ξ_n are known functions of n. We suppose that the homogeneous solution has been obtained in the form

$$x_n^{(H)} = \eta_1 \phi_n, \tag{605}$$

where η_1 is an arbitrary constant. The function ϕ_n necessarily satisfies

$$T\phi_n = 0. \tag{606}$$

Since the total solution to (604) is the sum of the homogeneous solution and any particular solution, we need to find a particular solution. We attempt to find a particular solution to (604) in the form

$$x_n^{(P)} = A_n \phi_n, \tag{607}$$

where A_n is an unknown function of n which is to be determined. Substituting (607) into (604) yields the condition

$$a_n A_n \phi_n + b_n A_{n-1} \phi_{n-1} = \xi_n, \tag{608}$$

which may be manipulated into the equivalent form

$$A_{n-1}(a_n\phi_n+b_n\phi_{n-1}) + a_n\phi_n(A_n-A_{n-1}) = \xi_n. \qquad (609)$$

Making use of (606), this expression can be further reduced to

$$a_n\phi_n\Delta A_{n-1} = \xi_n, \qquad (610)$$

so that

$$\Delta A_{n-1} = \frac{\xi_n}{a_n\phi_n} . \qquad (611)$$

In view of (598-603), we have

$$A_n = \eta_2 + \sum_{k=L}^{n} \frac{\xi_k}{a_k\phi_k} , \qquad (612)$$

where η_2 is an arbitrary constant. We chose $\eta_2 = 0$ since the inclusion of a nonzero η_2 would serve only to unnecessarily include part of the homogeneous solution to (604) in the particular solution to (604).*
Accordingly, we have

$$A_n = \sum_{k=L}^{n} \frac{\xi_k}{a_k\phi_k} . \qquad (613)$$

Substituting (613) into (607), we find

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*The reader may wish to carry a nonzero η_2 through the remainder of the computations. He will find that the choice $\eta_2 = 0$ is justified because any change in the value of η_2 can be counteracted by a change in the value of η_1.

$$x_n^{(P)} = \phi_n \sum_{k=L}^{n} \frac{\xi_k}{a_k \phi_k} . \tag{614}$$

Thus, the total solution obtained by this "method of variation of parameters" is

$$x_n = x_n^{(H)} + x_n^{(P)} = \eta_1 \phi_n + \phi_n \sum_{k=L}^{n} \frac{\xi_k}{a_k \phi_k} . \tag{615}$$

The General Linear Difference Equation of Second Order

The general difference equation of second order may be written in the form

$$Tx_n \equiv a_n x_{n+1} + b_n x_n + c_n x_{n-1} = \xi_n , \tag{616}$$

where T is a linear operator defined by the equation and a_n, b_n, c_n, and ξ_n are known functions of n. We presume that the homogeneous solution to (616) has been obtained in the form

$$x_n^{(H)} = \eta_1 \phi_n + \eta_2 \psi_n , \tag{617}$$

where η_1, η_2 are arbitrary constants and where

$$T\phi_n = T\psi_n = 0 \tag{618}$$

is satisfied. We seek to find a particular solution to (616) in the form

$$x_n^{(P)} = A_n\phi_n + B_n\psi_n, \tag{619}$$

where A_n and B_n are unknown functions of n to be determined.

From (619),

$$\begin{aligned} x_n^{(P)} &= A_n\phi_n + B_n\psi_n \\ &= A_{n-1}\phi_n + (A_n - A_{n-1})\phi_n \\ &\qquad + B_{n-1}\psi_n + (B_n - B_{n-1})\psi_n \\ &= A_{n-1}\phi_n + B_{n-1}\psi_n + [\phi_n \Delta A_{n-1} + \psi_n \Delta B_{n-1}]. \end{aligned} \tag{620}$$

Since *two* conditions will be required to determine both A_n and B_n--we now have only the condition (616)--we may arbitrarily impose the condition

$$\phi_n \Delta A_{n-1} + \psi_n \Delta B_{n-1} = 0. \tag{621}$$

Substituting (621) into (620) yields

$$x_n^{(P)} = A_{n-1}\phi_n + B_{n-1}\psi_n. \tag{622}$$

Again from (619),

$$x_{n+1}^{(P)} = A_{n+1}\phi_{n+1} + B_{n+1}\psi_{n+1}$$

$$= A_{n-1}\phi_{n+1} + B_{n-1}\psi_{n+1}$$

$$+ (A_{n+1}-A_n)\phi_{n+1} + (B_{n+1}-B_n)\psi_{n+1}$$

$$+ (A_n-A_{n-1})\phi_{n+1} + (B_n-B_{n-1})\psi_{n+1}$$

$$= A_{n-1}\phi_{n+1} + B_{n-1}\psi_{n+1} + [\phi_{n+1}\Delta A_n+\psi_{n+1}\Delta B_n]$$

$$+ \phi_{n+1}\Delta A_{n-1} + \psi_{n+1}\Delta B_{n-1}. \qquad (623)$$

The bracketed expression vanishes by virtue of Equation (621) and hence,

$$x_{n+1}^{(P)} = A_{n-1}\phi_{n+1} + B_{n-1}\psi_{n+1} + \phi_{n+1}\Delta A_{n-1} + \psi_{n+1}\Delta B_{n-1}. \qquad (624)$$

Finally, from (619),

$$x_{n-1}^{(P)} = A_{n-1}\phi_{n-1} + B_{n-1}\psi_{n-1}. \qquad (625)$$

Now substituting Equations (622,624-625) into (616) and making use of (618) yields

$$\xi_n = a_n(A_{n-1}\phi_{n+1} + B_{n-1}\psi_{n+1} + \phi_{n+1}\Delta A_{n-1} + \psi_{n+1}\Delta B_{n-1})$$

$$+ b_n(A_{n-1}\phi_n + B_{n-1}\psi_n) + c_n(A_{n-1}\phi_{n-1} + B_{n-1}\psi_{n-1})$$

$$= A_{n-1}(a_n\phi_{n+1} + b_n\phi_n + c_n\phi_{n-1})$$

$$+ B_{n-1}(a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1})$$

$$+ a_n\phi_{n+1}\Delta A_{n-1} + a_n\psi_{n+1}\Delta B_{n-1}$$

$$= a_n(\phi_{n+1}\Delta A_{n-1} + \psi_{n+1}\Delta B_{n-1}), \tag{626}$$

The two conditions (621,626) thus imply

$$\Delta A_{n-1} = \frac{\begin{vmatrix} 0 & \psi_n \\ \xi_n & a_n\psi_{n+1} \end{vmatrix}}{\begin{vmatrix} \phi_n & \psi_n \\ a_n\phi_{n+1} & a_n\psi_{n+1} \end{vmatrix}} = \frac{-\xi_n\psi_n}{a_n(\phi_n\psi_{n+1}-\phi_{n+1}\psi_n)} , \tag{627}$$

$$\Delta B_{n-1} = \frac{\begin{vmatrix} \phi_n & 0 \\ a_n\phi_{n+1} & \xi_n \end{vmatrix}}{\begin{vmatrix} \phi_n & \psi_n \\ a_n\phi_{n+1} & a_n\psi_{n+1} \end{vmatrix}} = \frac{\xi_n\phi_n}{a_n(\phi_n\psi_{n+1}-\phi_{n+1}\psi_n)} . \tag{628}$$

Hence, using (598-603), we have*

$$A_n = -\sum_{k=L}^{n} \frac{\xi_k \psi_k}{a_k(\phi_k \psi_{k+1} - \phi_{k+1}\psi_k)} , \tag{629}$$

$$B_n = \sum_{k=L}^{n} \frac{\xi_k \phi_k}{a_k(\phi_k \psi_{k+1} - \phi_{k+1}\psi_k)} , \tag{630}$$

and

$$x_n^{(P)} = A_n \phi_n + B_n \psi_n = \sum_{k=L}^{n} \frac{\xi_k(\phi_k \psi_n - \phi_n \psi_k)}{a_k(\phi_k \psi_{k+1} - \phi_{k+1}\psi_k)}, \tag{631}$$

where L may be arbitrarily selected in accordance with the previous discussion.

Since for *any* particular solution $x_n^{(P)}$,

$$x_n = x_n^{(H)} + x_n^{(P)}, \tag{632}$$

we have as the general solution to Equations (616),

$$x_n = \eta_1 \phi_n + \eta_2 \psi_n + \sum_{k=L}^{n} \frac{\xi_k(\phi_k \psi_n - \phi_n \psi_k)}{a_k(\phi_k \psi_{k+1} - \phi_{k+1}\psi_k)} . \tag{633}$$

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*Here the homogeneous parts of A_n and B_n have been chosen to be zero. See preceding footnote.

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"Refinement of Aircraft Reworks Costing in the Navy Resources Model," *(INS) 1200-72*, Center for Naval Analyses, Arlington, Va., August 1972, ii + 15 pp.;

as well as a number of short papers and memoranda on strategic mobility, strategic deployment, aircraft reworks, and pilot training.